UNIVERSITY OF BIRMINGHAM | GRAVITATIONAL WAVE ASTRONOMY

LECTURE 1: INTRODUCTION TO GRAVITATIONAL WAVES

PATRICIA SCHMIDT

ISCRA 2024

WHO AM I?

- **Patricia Schmidt**: Associate Professor in Gravitational Waves, University of Birmingham, UK
  - Member of LIGO since 2010
  - Co-chair of the **parameter estimation** technical working group in LIGO since 2020
  - Co-chair of the **waveforms** division of the Observational Science Board of the Einstein Telescope Collaboration since 2021
- <u>Selected research interests:</u>
  - Modelling gravitational waves from black holes & neutron stars incl. numerical relativity
  - GW source characterisation (parameter estimation)
  - Measuring black hole spins
  - Constraining the neutron equation of state with GWs

"Old Joe" - Tolkien's inspiration for the Tower of Sauron

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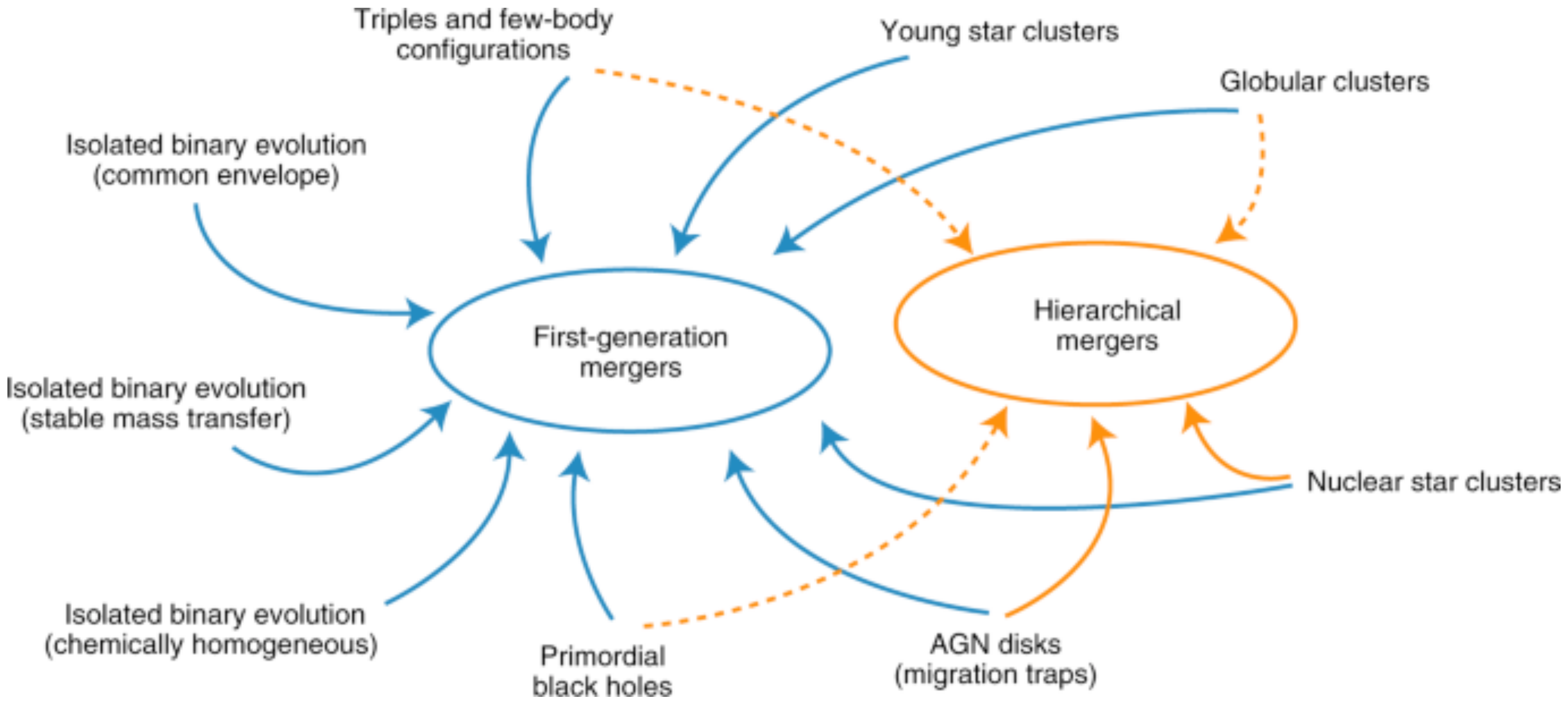
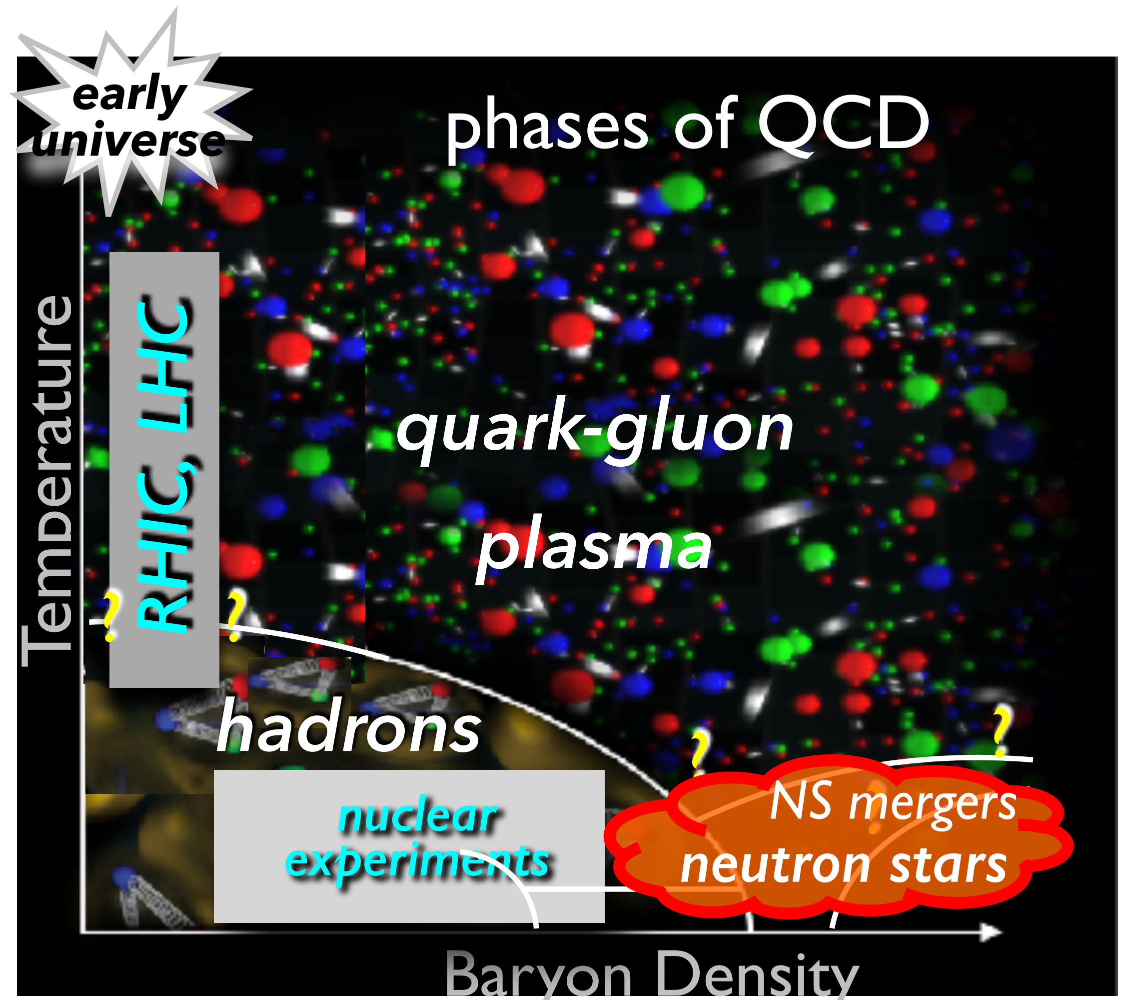
GRAVITATIONAL-WAVE SCIENCE

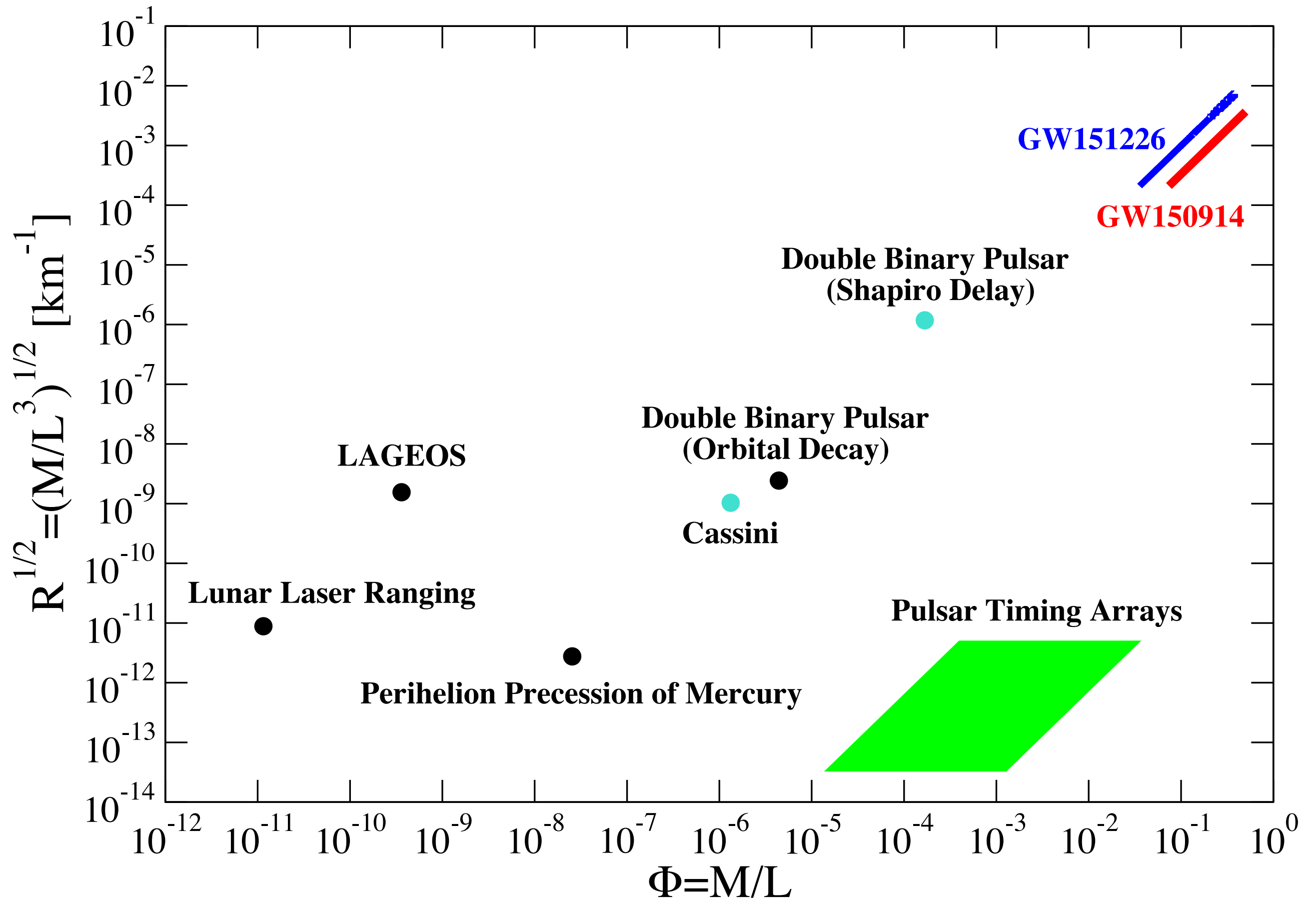
- Discover the **dark side** of the Universe
  - Detect and determine **properties** of astrophysical (and primordial?) black holes
  - Measure **merger rates** of compact binaries
  - Inform **binary formation** models
- Infer the **EOS of matter** at supra-nuclear densities, e.g. in neutron stars
- **Test GR** in the strong-field, high-curvature regime
- Independently measure the **expansion rate** of the Universe
- **Multimessenger astrophysics**
- Dark energy, dark matter
- …



[Gerosa&Fishbach]



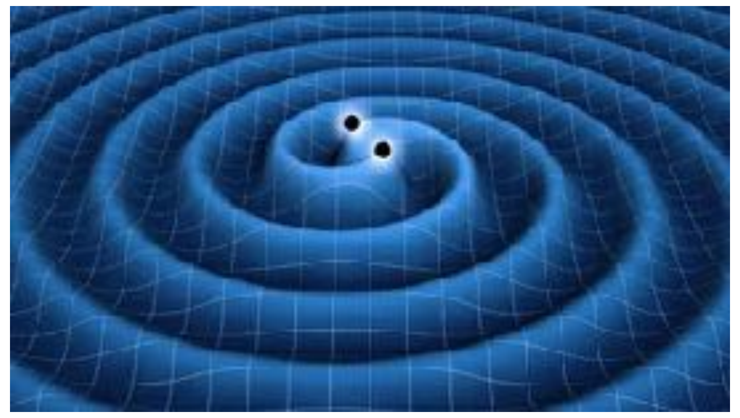
Credit: F. Linde

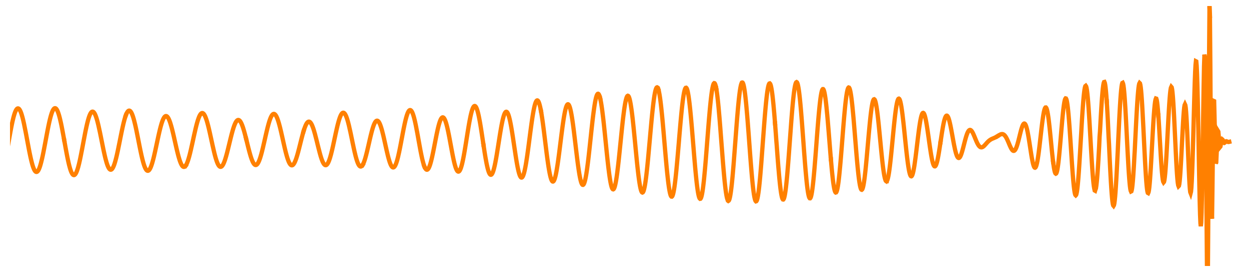


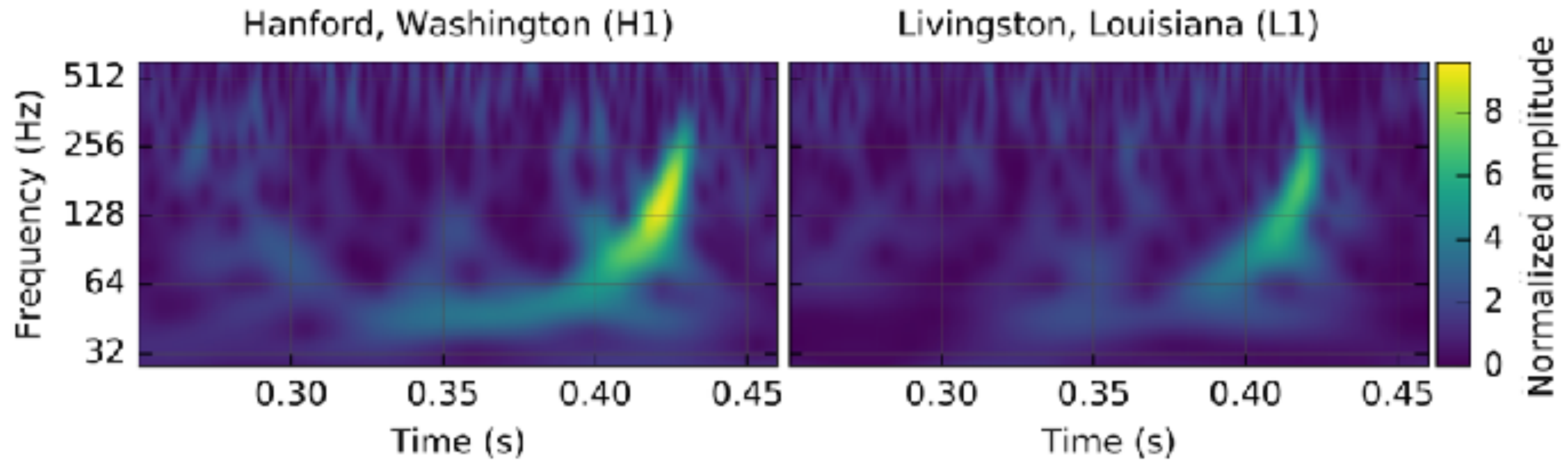
[Yunes, 2016]

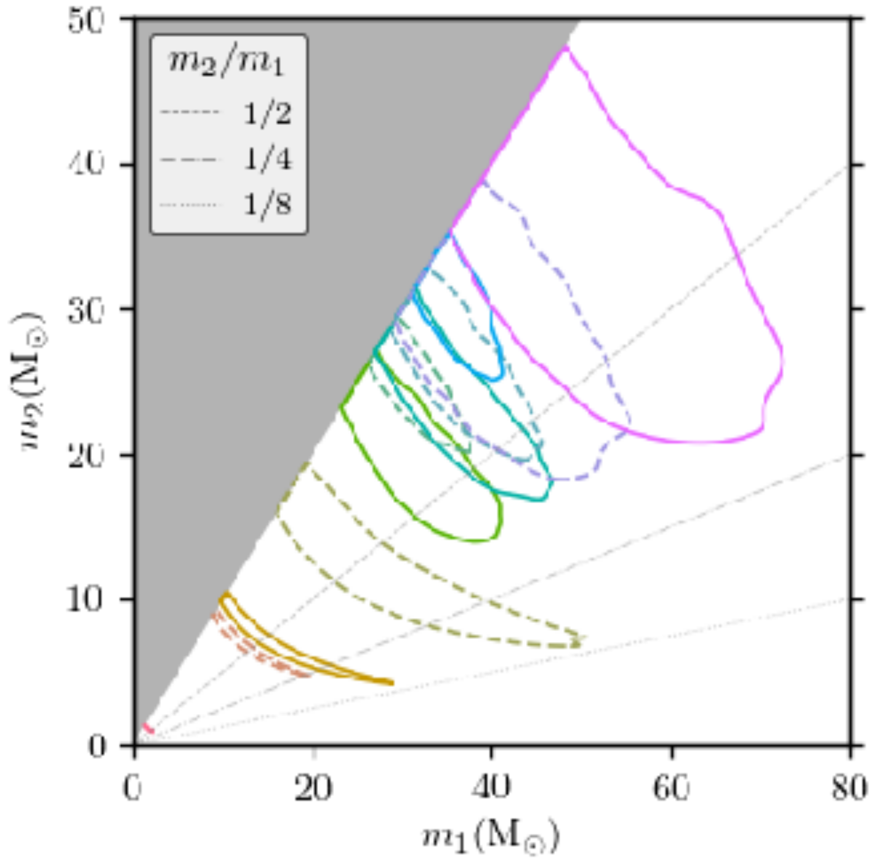
OUTLINE

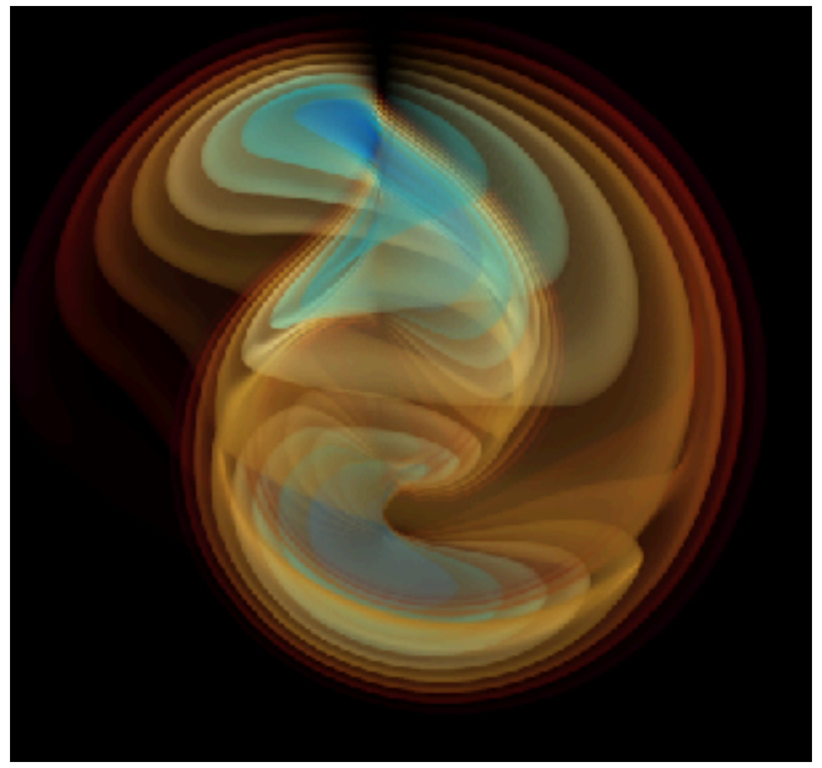
- **Lecture 1: Introduction; 22.7. @ 10:05**
  - What are gravitational waves?
  - Sources of gravitational waves
  - Gravitational-wave detectors
- **Lecture 2: Data Analysis for compact binaries; 23.7. @ 16:05**
  - Detection: Matched filtering
  - Parameter Estimation
  - Modelling gravitational waves from compact binaries
- **Lecture 3: Observations; 27.7. @ 15:00**
  - Gravitational-wave observations to date
  - Future missions and prospects









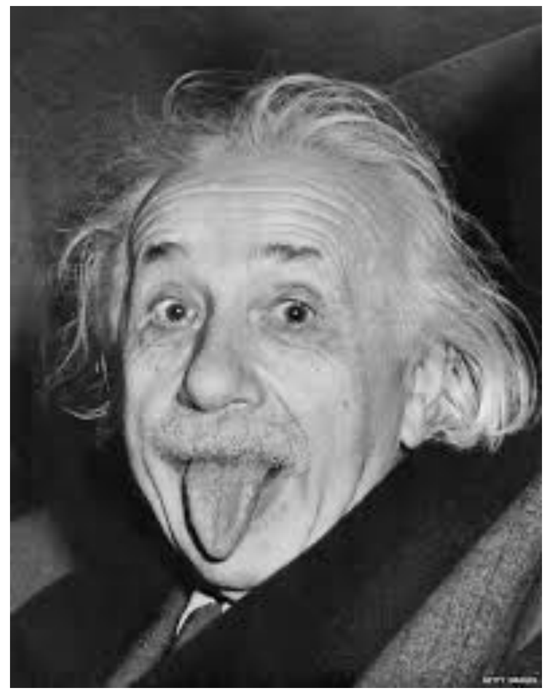


GRAVITATIONAL WAVES IN A NUTSHELL

- Recommended literature:
    - Michele Maggiore, “Gravitational Waves”, Volume 1
    - Bernard Schutz, “A first course in General Relativity”

GR REMINDER

- The gravitational field is a geometric property of 4D spacetime: **curvature**
  - **Metric tensor** $g_{\mu\nu}$: how to measure distances and angles in a curved manifold
  - Mass/energy curve spacetime



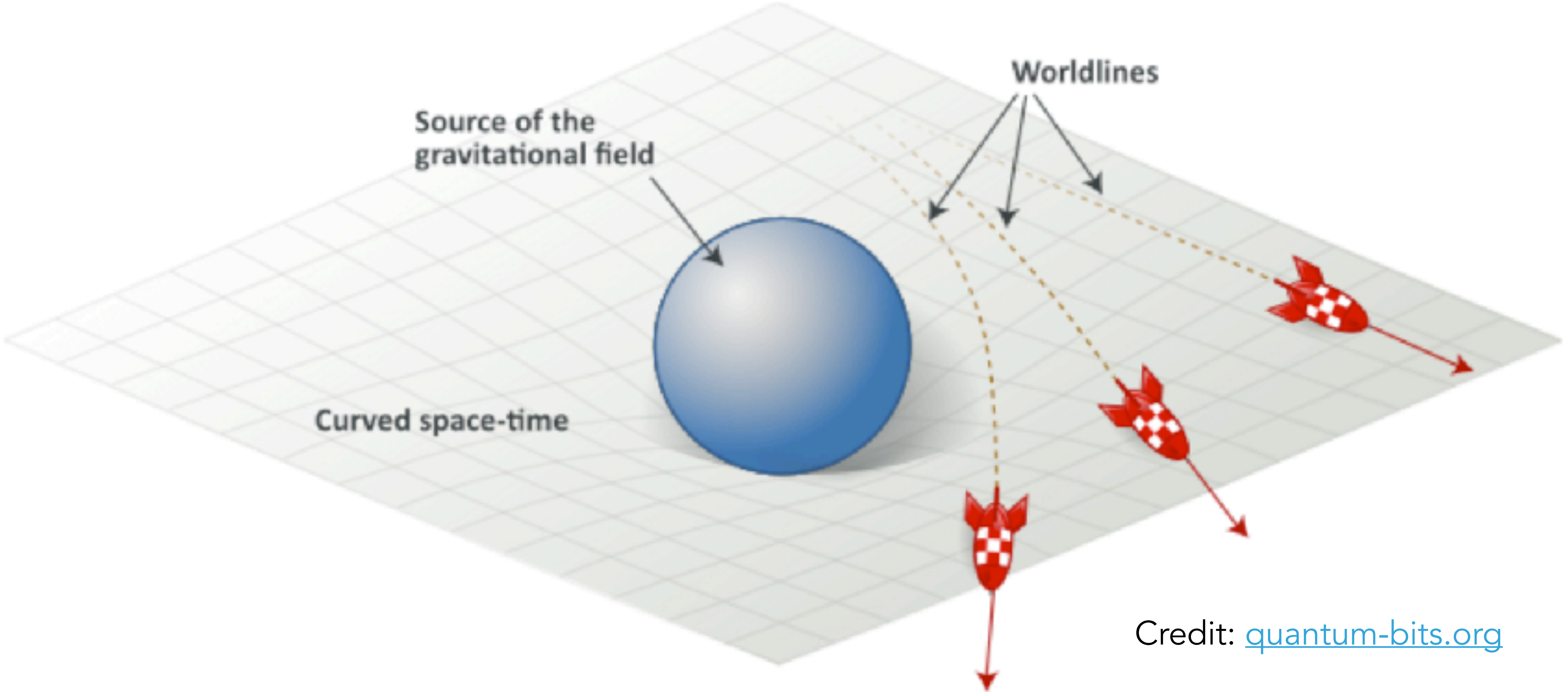
Conventions:

$$\mathrm{sign}(\eta_{\mu\nu}) = (-1,1,1,1)$$
$$u^\mu v_\mu = \sum_\mu u^\mu v_\mu$$
$$\mu \in \{0,1,2,3\}$$
$$i \in \{1,2,3\}$$
$$G = c = 1$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Einstein field equations

- Locally, for freely-falling observers the laws of special relativity hold (equivalence principle)
  - Freely-falling observers move along **geodesics** (shortest paths in general manifolds)
  - **Tidal effects** determine the relative acceleration between 2 freely-falling observers



Credit: quantum-bits.org

LINEARISED GRAVITY

- GWs are a fundamental prediction of General Relativity (GR): propagating oscillations of the gravitation field generated by **accelerating masses**
  - Transverse waves travelling at the speed of light c
- Let us consider the vacuum **Einstein field equations** (far away from the source of the gravitational field):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

- **Linearised gravity:** Far away from the source of the gravitational field, the metric $g_{\mu\nu}$ is that of flat Minkowski space with a small metric perturbation $h_{\mu\nu}$, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad ||h_{\mu\nu}|| \ll 1$$

  - Compute all relevant quantities keeping only the terms linear in $h_{\mu\nu}$ (higher order terms are discarded)
  - Work with the trace-reversed metric perturbation to simplify expressions: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

LINEARISED GRAVITY

- Make use of the gauge freedom in GR!
  - Using the **Lorenz (harmonic) gauge**, $\partial^\nu \bar{h}_{\mu\nu} = 0$, the Einstein field equations reduce to a wave equation for the trace-reversed metric perturbation tensor:

$$\Box \bar{h}_{\mu\nu} \equiv \eta_{\mu\nu}\partial^\mu \partial^\nu \bar{h}_{\mu\nu} = \left(-\frac{1}{c^2}\partial_t^2 + \nabla^2 \right) \bar{h}_{\mu\nu} = 0$$

flat-space d'Alembertian

- Solutions to the wave equation are (superpositions of) plane waves:

$$\bar{h}_{\mu\nu}(t;\vec{x}) = \mathrm{Re} \int d^3k A_{\mu\nu}(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

  - Note: $k^\mu = (\omega, \vec{k})$ and $k^\mu A_{\mu\nu} = 0$ because of the Lorenz gauge

These are gravitational waves!

TRANSVERSE-TRACELESS GAUGE

- The Lorenz gauge condition does not fix the GR gauge freedom completely for globally vacuum, asymptotically flat spacetimes
  - Impose **4 additional gauge conditions**: $h = 0$ (traceless) & $h_{00} = h_{0i} = 0$ (purely spatial)
  - From the Lorenz gauge condition it follows that $\partial^i h_{ij} = 0$, i.e. the metric perturbation is transverse
  - This is the transverse-traceless (TT) gauge, which is not necessary but very convenient.
  - The remaining DOF contain only physical information, non-gauge information about GWs!
- For a plane-wave travelling along the z-axis, the metric perturbation tensor in the TT gauge becomes:

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos(\omega(t - z/c))$$

- 2 DOF: $h_+, h_\times$ are the two independent gravitational-wave polarisations
- Note: One can show that the radiative DOF are always contained in the TT-part of the metric perturbation in any gauge!

INTERACTION OF GWS WITH TEST MASSES

- In curved space, test masses move along **geodesics** parameterised by the **proper time** τ:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

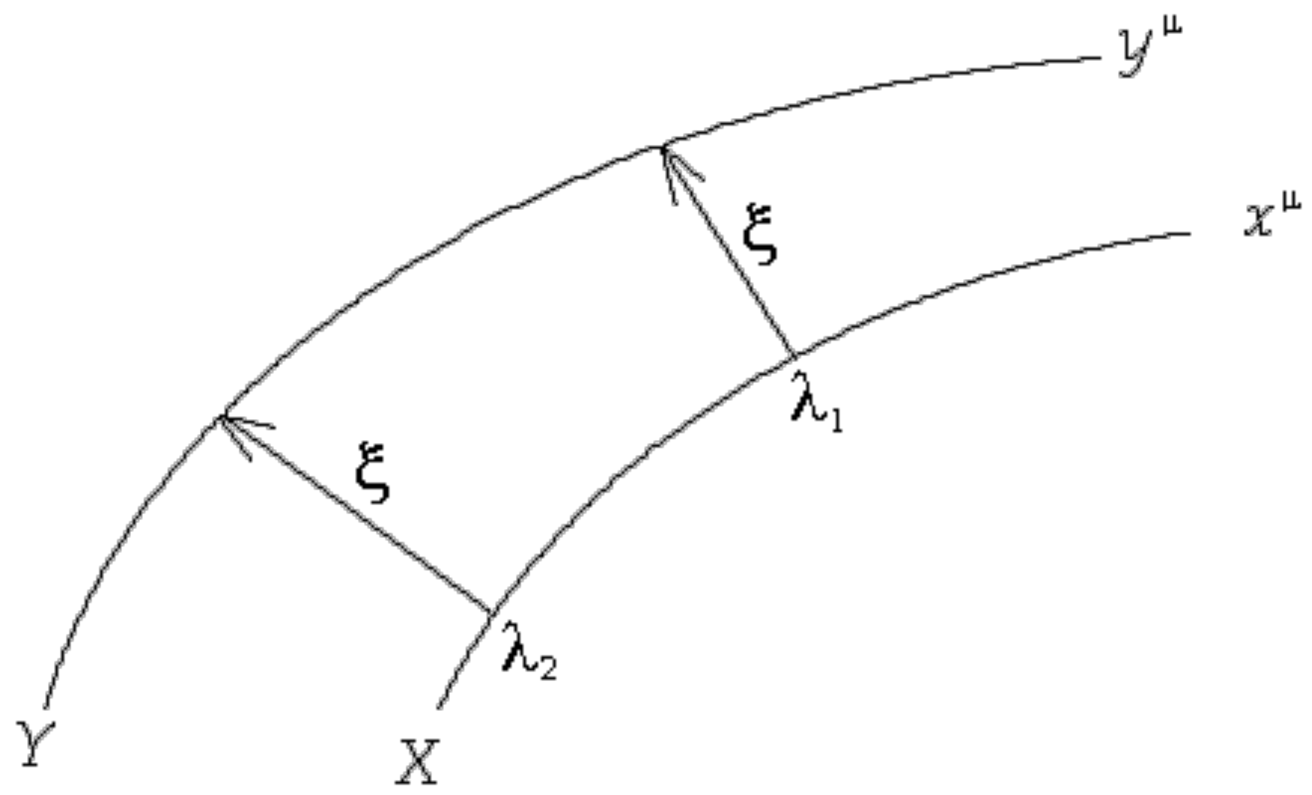
geodesic equation

- Let us consider two nearby geodesics, separated by an infinitesimal vector $\xi^\mu(\tau)$. If the separation is much smaller than the typical scale of the variation of the gravitational field, the first-order expansion leads to the **geodesic deviation equation**:

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\sigma}\frac{dx^\mu}{d\tau}\frac{d\xi^\sigma}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} = 0$$

covariant derivative

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\xi^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$



- The **separation between the two geodesics** changes with time in the presence of a gravitational field
  - Two nearby time-like geodesics experience a tidal force, which is determined by the Riemann tensor.

INTERACTION OF GWS WITH TEST MASSES

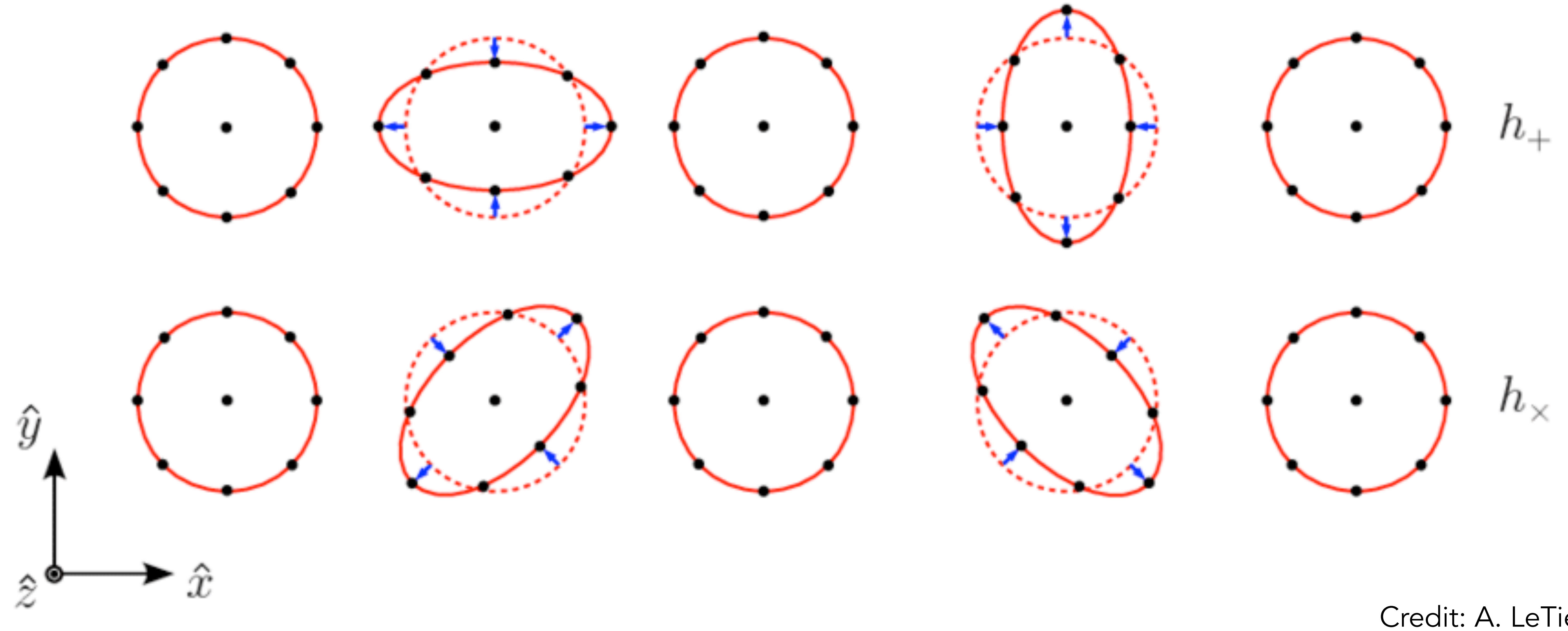
- Consider a local rest frame at a point P; i.e.
  $g_{\mu\nu}(P) = \eta_{\mu\nu} \quad \rightarrow \quad \Gamma^{\mu}{}_{\nu\sigma}(P) = 0$

- Consider a non-relativistic observer (e.g. a GW detector), then $dx^i/d\tau \ll dx^0/d\tau$

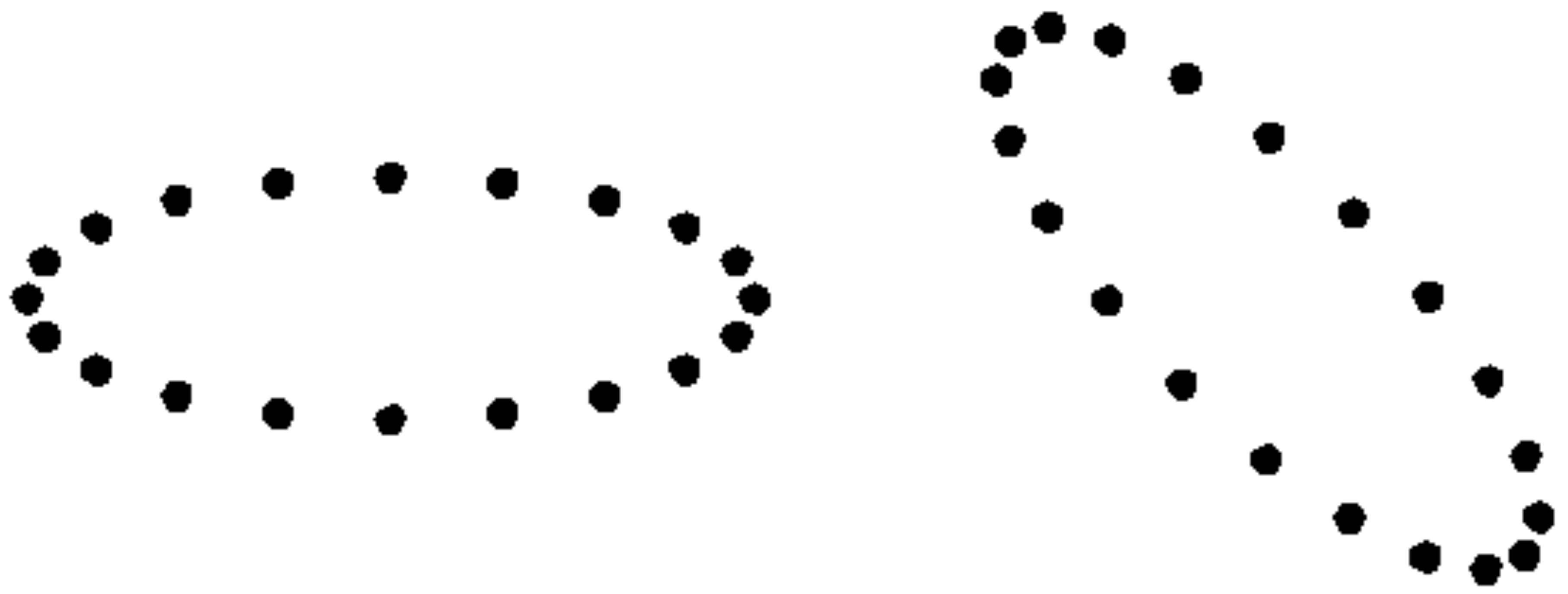
- Under these assumptions, the geodesic deviation equation reduces to:

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\xi^j$$

- Gravitational waves have the effect of tidal waves, i.e. they change the **proper separation** between two freely-falling test masses periodically: "stretching" and "squeezing" of spacetime



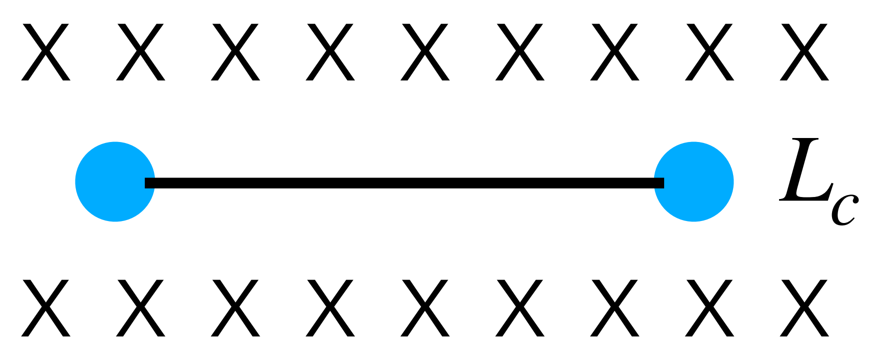
Credit: A. LeTiec



INTERACTION OF GWS WITH TEST MASSES

- Let us consider two freely falling test masses located at $z = 0$ and separated by a **coordinate distance** L_c along the x-axis.

- Consider a GW travelling down the z-axis in the TT gauge: $h_{\mu\nu}^{\mathrm{TT}}(t; z)$.



- Then the **proper distance** L between the two test masses is given by:

$$L = \int_0^{L_c} dx \sqrt{g_{xx}} = \int_0^{L_c} dx \sqrt{1 + h_{xx}^{\mathrm{TT}}(t; z = 0)}$$

$$\simeq \int_0^{L_c} dx \left[1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}(t; z = 0)\right] = L_c \left[1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}(t; z = 0)\right]$$

  - Note: We used the fact that the coordinate separation remains fixed in the TT gauge.

- When a GW passes, the proper separation changes by a fractional length change (**strain**) $\delta L / L$ given by

$$\frac{\delta L}{L} \simeq \frac{1}{2} h_{xx}^{\mathrm{TT}}(t; z = 0)$$

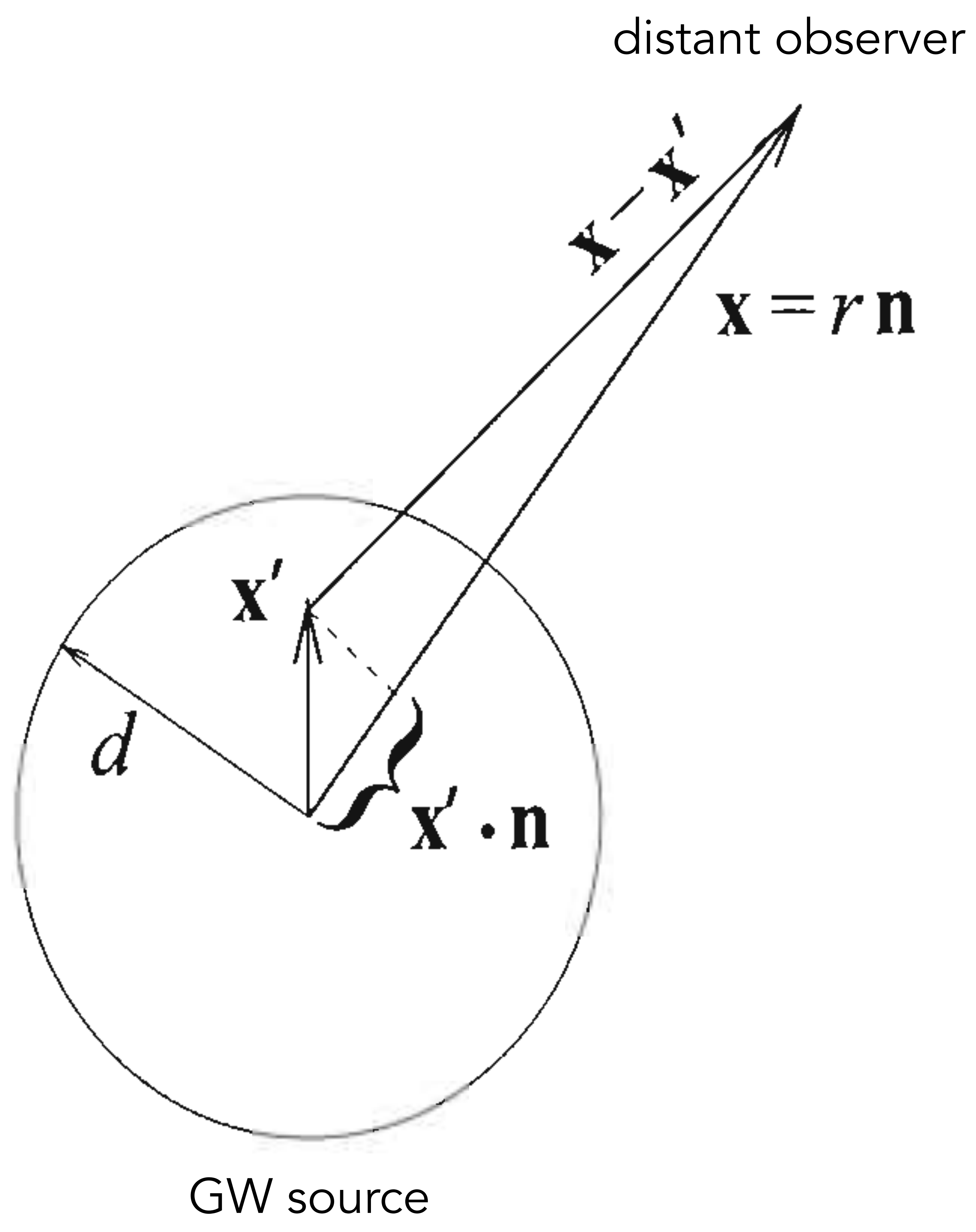
This fractional length change = strain is what we measure in GW detectors!

GENERATION OF GRAVITATIONAL WAVES: QUADRUPOLE FORMULA

- Let us assume a **slowly moving source** in linearised gravity: $v \ll c$
- The solutions to the **inhomogeneous wave equation** are plane waves (in the Lorenz gauge):

$$\bar{h}_{\mu\nu}(t;\vec{x}) = 4\int d^3x' \frac{T_{\mu\nu}(t-|\vec{x}-\vec{x}'|;\vec{x}')}{|\vec{x}-\vec{x}'|}$$

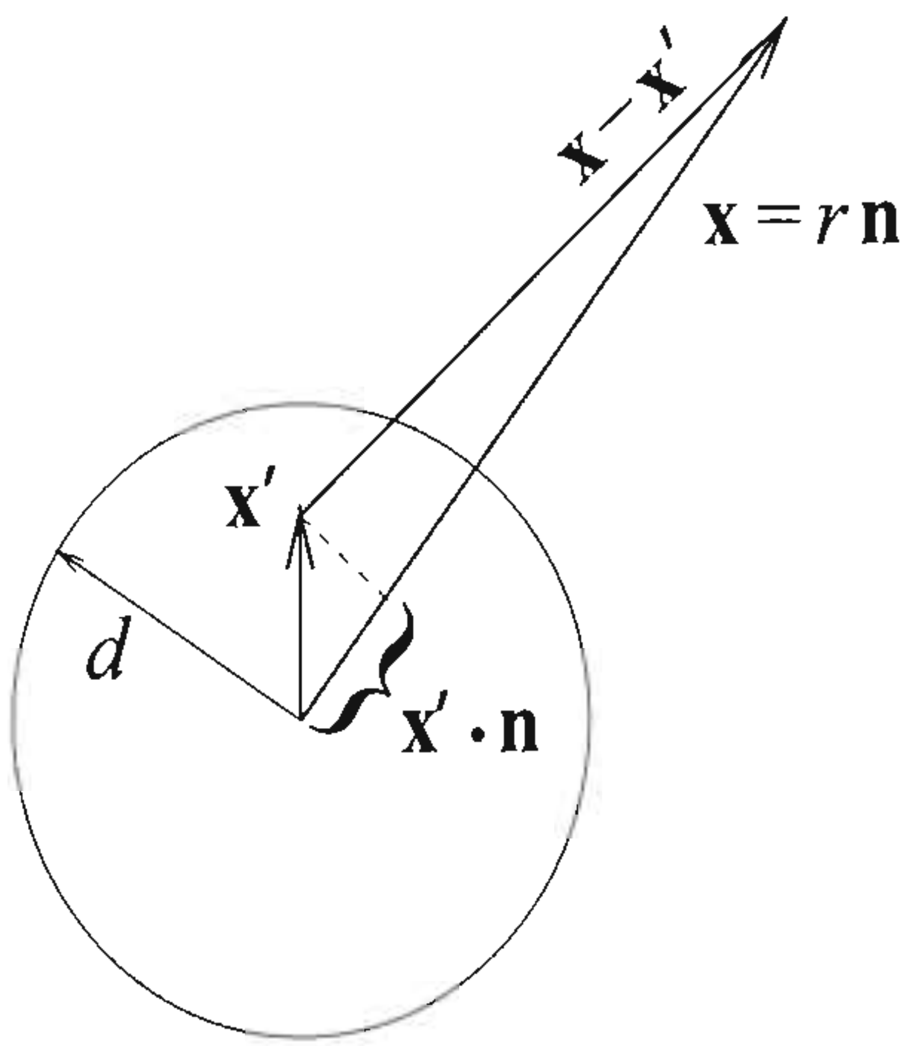
  - Recall that the radiative degrees of freedom are contained in the spatial TT-part of the metric: $\mu\nu \rightarrow ij$
- At large distance from the source, we can perform a **multipole expansion** of the denominator analogous the EM to find $\bar{h}_{ij}(t;\vec{x}) = \frac{4}{r}\int d^3x' T_{ij}(t-r;\vec{x}')$ (at linear order), where $r := |\vec{x}|$.



GENERATION OF GRAVITATIONAL WAVES: QUADRUPOLE FORMULA

- Using the **continuity equation** in linearised gravity, i.e. $\partial_\mu T^{\mu\nu} = 0$, we can further simplify this integral:

$$\frac{4}{r}\int d^3x' T_{ij}(t-r;\vec{x}') = \frac{2}{r}\frac{\partial^2}{\partial t^2}\int d^3x' \rho x'^i x'^j$$



- Using the definition of the moment of inertia tensor, we arrive at:

$$\bar{h}_{ij}(t;\vec{x}) = \frac{2}{r}\frac{d^2 I_{ij}(t-r)}{dt^2}$$

- By projecting out the TT part, we arrive at the final answer - the **quadrupole formula**:

$$h_{ij}^{\mathrm{TT}}(t;\vec{x}) = \frac{2}{r}\frac{d^2\mathcal{I}_{kl}(t-r)}{dt^2}P_{ik}(\hat{n})P_{jl}(\hat{n})$$

mass quadrupole

A time-varying quadrupole moment sources GWs!

GENERATION OF GRAVITATIONAL WAVES: LUMINOSITY

- Gravitational waves are some of the most luminous events in the universe
  - GW150914 emitted about 3 solar masses in GWs in 0.2s! ($\approx 3.6 \times 10^{49}\mathrm{W}$)
- GW waves **carry energy** and **momentum** away from the source
- The **stress-energy tensor of a propagating gravitational field** is given by the Isaacson expression

$$T_{\mu\nu} = \frac{1}{32\pi}\langle h^{\mathrm{TT}}_{jk,\mu} h^{\mathrm{TT}jk}_{,\nu}\rangle$$

  - Brackets denote an average of regions of the size of the wavelength and times of the length of the period.
- The **GW luminosity** is obtained by integrating the flux over a distant sphere:

$$L_{\mathrm{GW}} = \frac{1}{5}\left(\sum_{j,k} \dddot{I}_{jk}\dddot{I}_{jk} - \frac{1}{3}\dddot{I}^{2}\right)$$

- Note: L_{GW} is dimensionless in geometric units but can be converted via the scale factor $L_0 = c^5/G = 3.6 \times 10^{52}\mathrm{W}$.
  - For comparison: GRB 221009A $\approx 10^{47}\mathrm{W}$

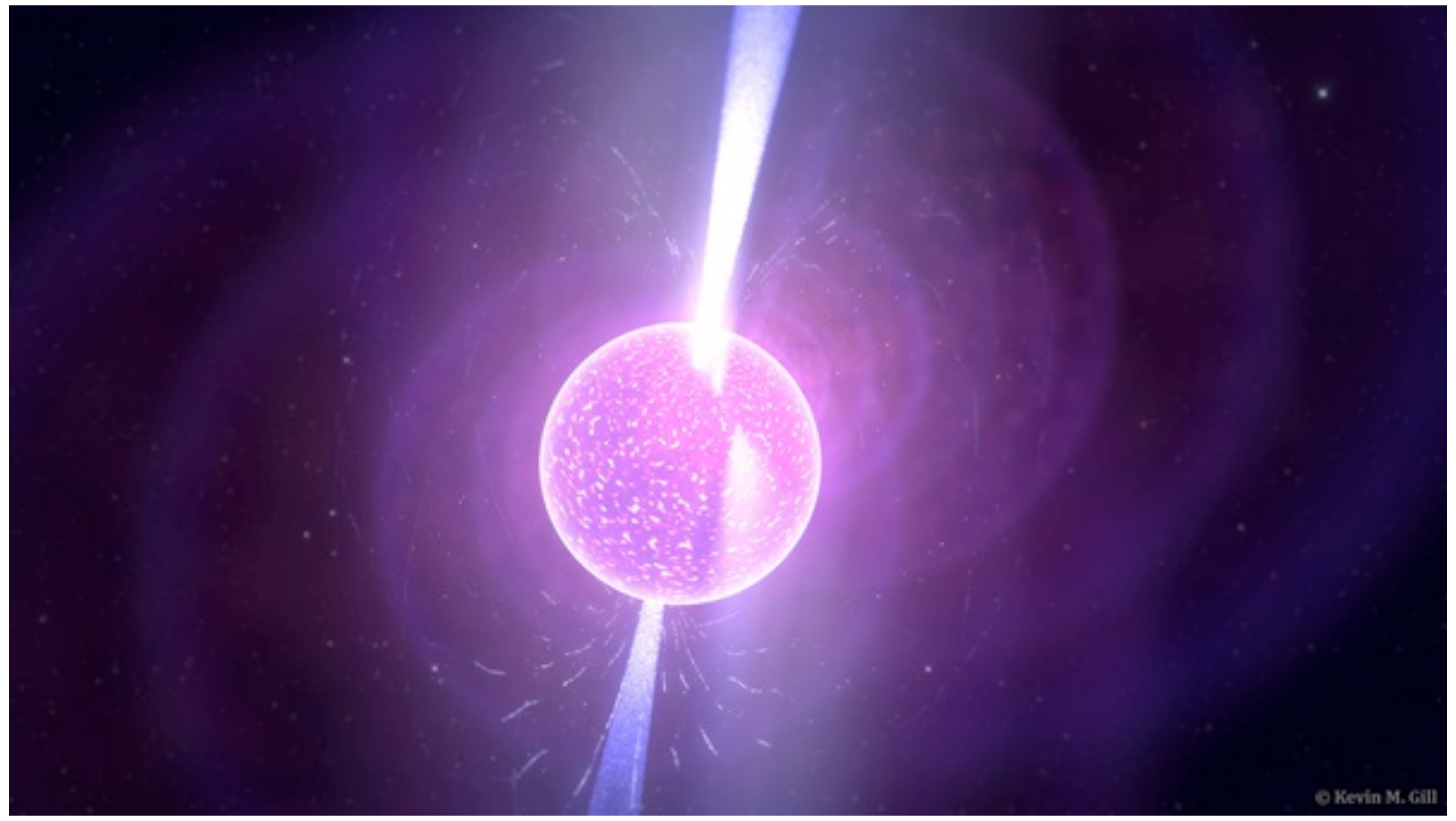
GRAVITATIONAL-WAVE SOURCES

- Recommended literature:
  - Sathyaprakash & Schutz, Living Reviews in Relativity

ASTROPHYSICAL SOURCES OF GRAVITATIONAL WAVES

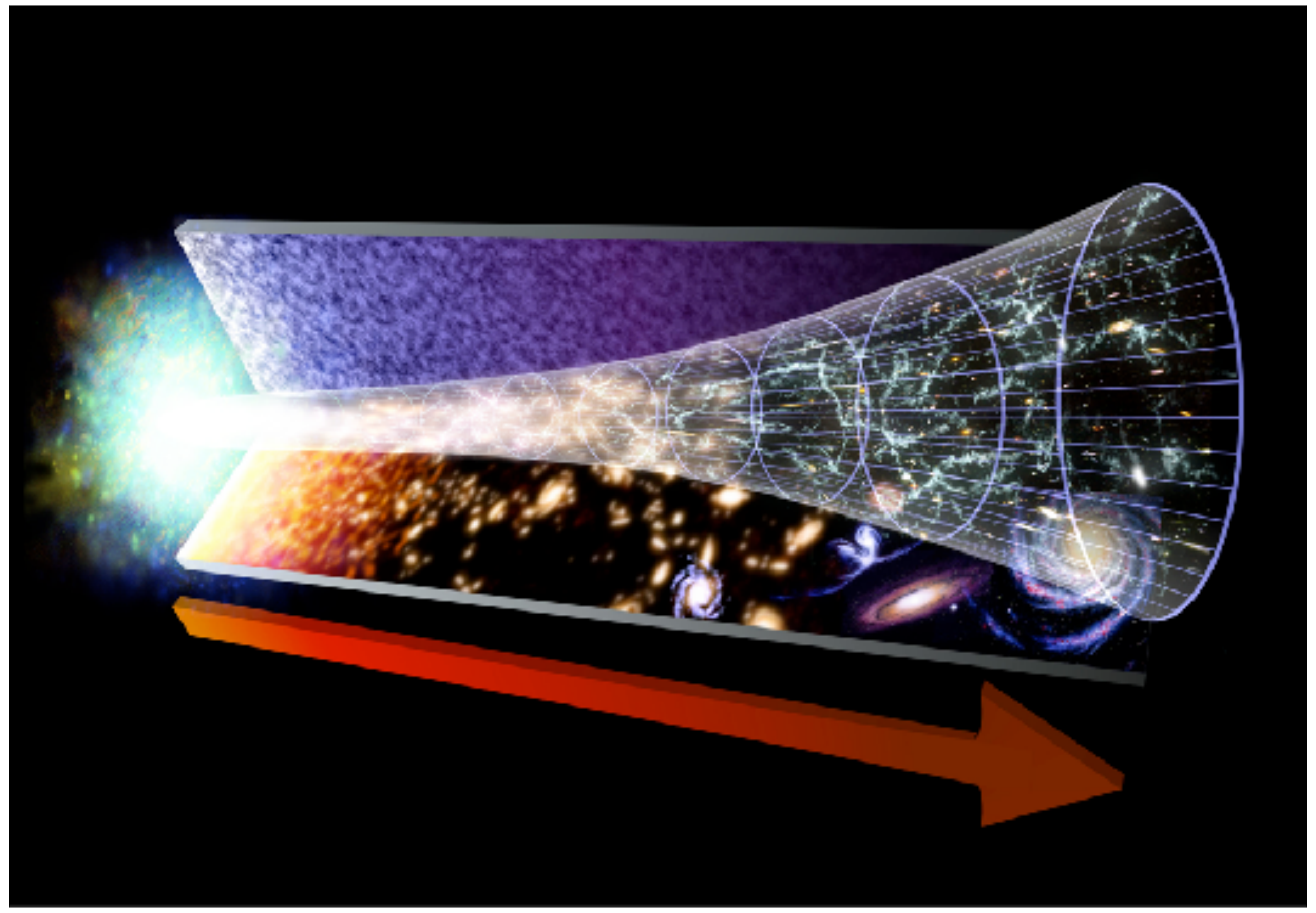
- Any mass distribution with a time-varying quadrupole moment sources gravitational waves

compact binaries



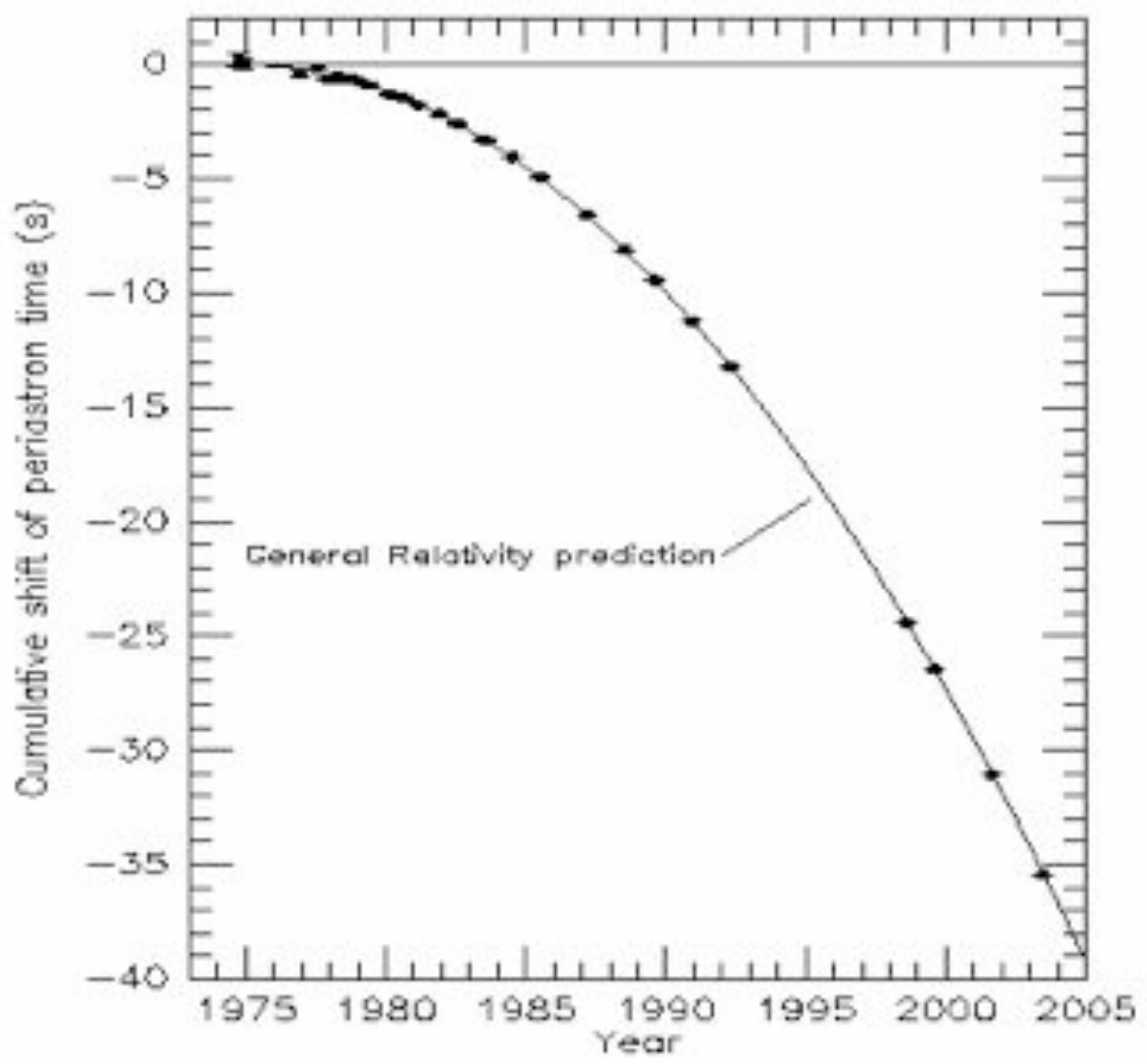
spinning neutron stars, pulsars, magnetars

supernovae



stochastic GW background (cosmological & astrophysical)

COMPACT BINARIES

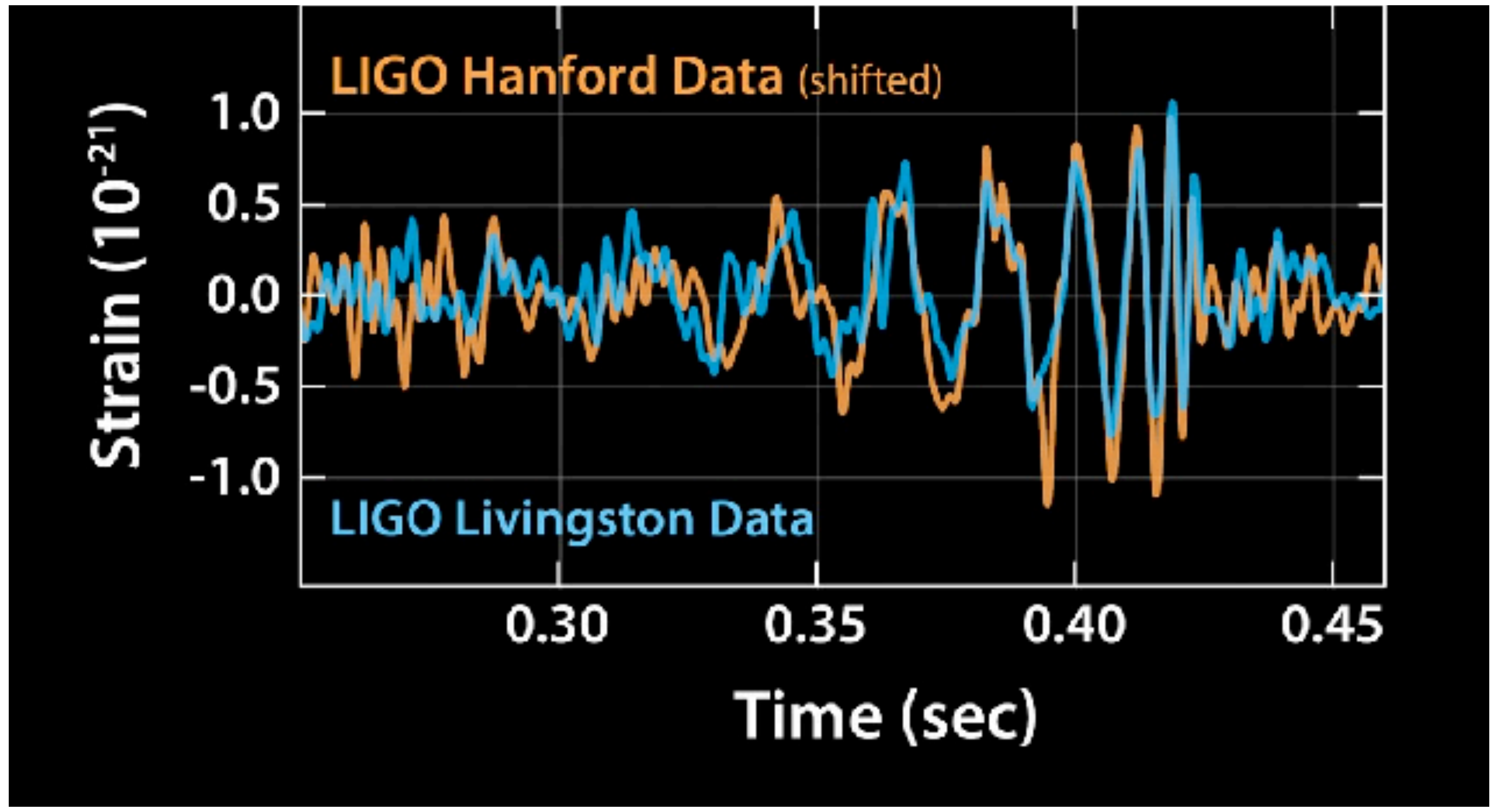


- Binary systems composed of **black holes** and **neutron stars** (also white dwarfs, supermassive black holes)
- Their orbital evolution is driven by the emission of gravitational waves, causing the orbit to shrink: **"chirp"** signal
- The GW amplitude of a compact binary can be estimated as

$$h \sim \frac{2}{r}\mathcal{M}_c^{5/3}\omega_{\rm orb}^{2/3} \qquad [\text{"chirp mass"}: \mathcal{M}_c = (m_1 m_2)^{3/5}/(m_1+m_2)^{1/5}]$$

- The **characteristic frequency** of a compact object can be estimated as

$$f_0 \sim \frac{1}{4\pi}\left(\frac{3M}{R^3}\right)^{1/2} \sim 1\,\mathrm{kHz}\left(\frac{10M_\odot}{M}\right)$$

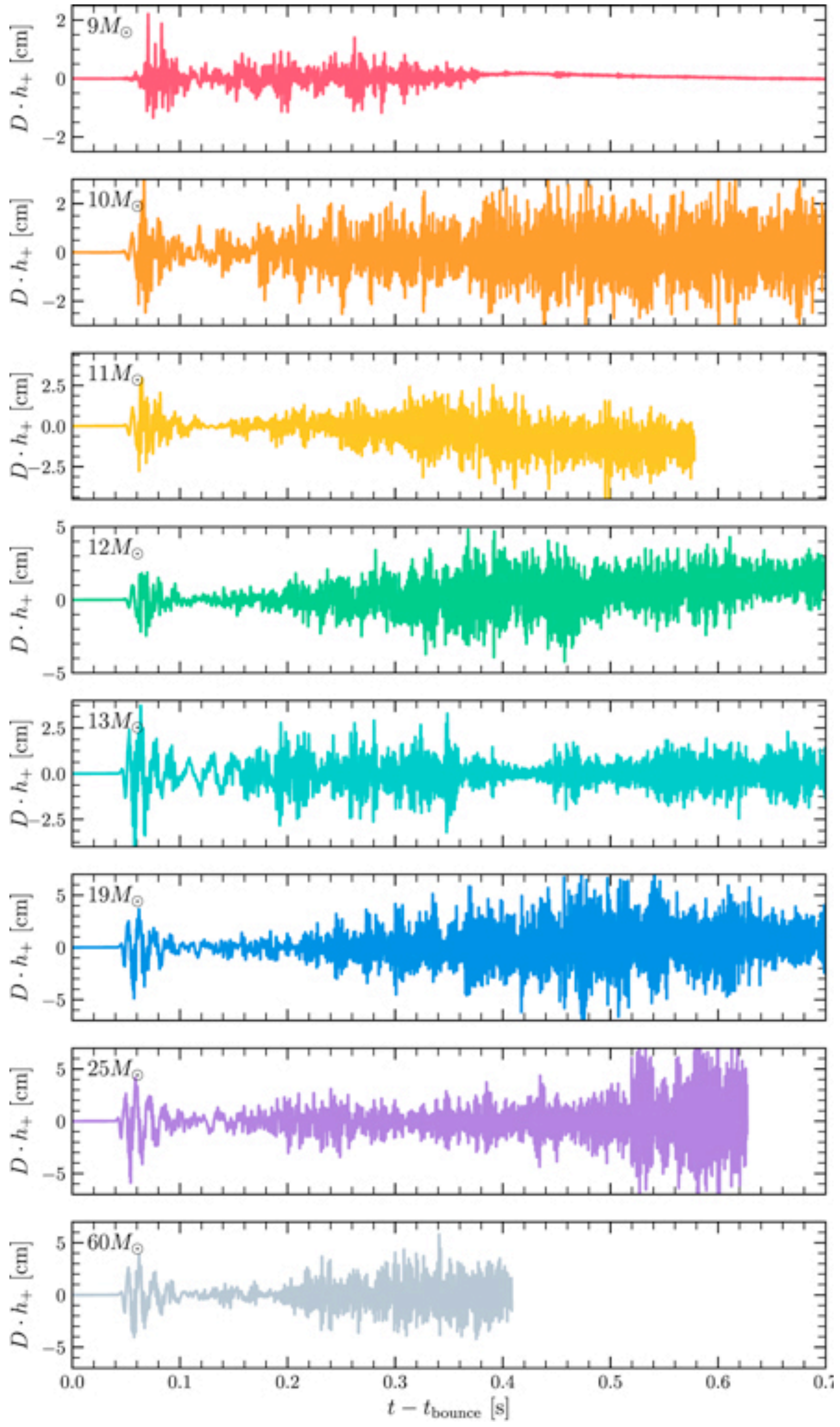


- Famous examples: Hulse-Taylor binary pulsar PSR B1913+16, GW150914, GW170817
  - Note: All directly GWs detected to date are consistent with compact binary mergers

CORE-COLLAPSE SUPERNOVAE

- Type II supernovae (CCSNe): Massive stars ($8M_\odot \lesssim M \lesssim 50M_\odot$) collapse at the end of their life and form either a black hole or a neutron star (remnant)
- If the collapse is non-spherical, GWs can carry away binding energy and angular momentum
- The Type II SNe rate in a Milky Way-like galaxy is 0.01-0.1 per year
- The GW amplitude can be estimated to be

$$h \sim 6 \times 10^{-21} \left(\frac{E_{\rm GW}}{10^{-7} M_\odot} \right)^{1/2} \left(\frac{1 \rm ms}{T} \right)^{1/2} \left(\frac{1 \rm kHz}{f} \right) \left(\frac{10 \rm kpc}{D_L} \right)$$



[Radice+, 2019]

ISOLATED NEUTRON STARS

- Gravitational pulsars = rotating neutron stars with asymmetry (" neutron star mountain")

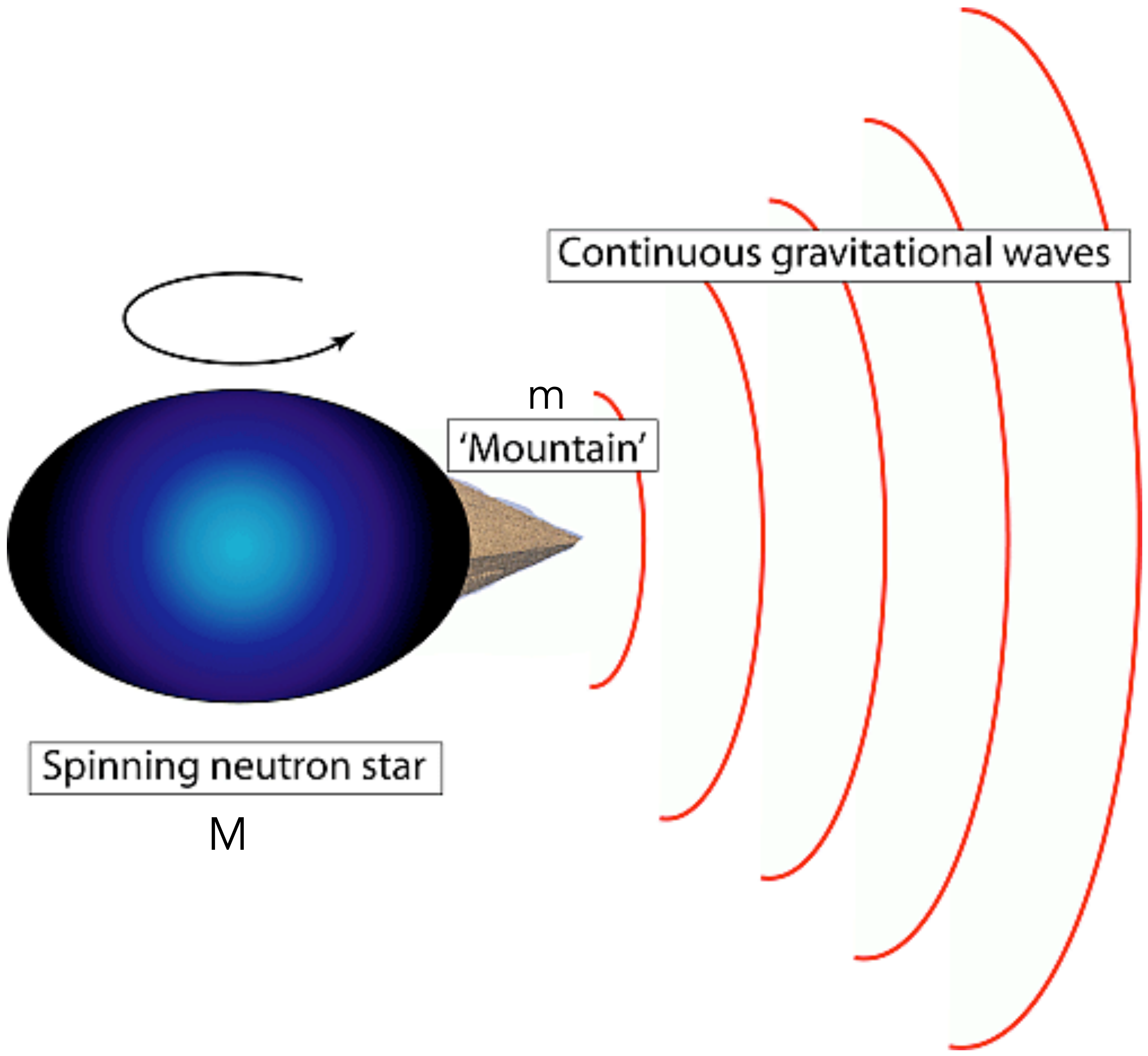
- The asymmetry leads to a non-symmetric quadrupole tensor

  - Assume a star with uniform density. Its moment of inertia is given by $I = 2MR^2/5$. A mountain with mass m will introduce a fractional asymmetry

$$\epsilon = \frac{5m}{2M}$$

  - As the star rotates, the mountain will emit GWs, causing the star to spin-down.

- Note: non-observation allows to set an upper limit on ϵ.

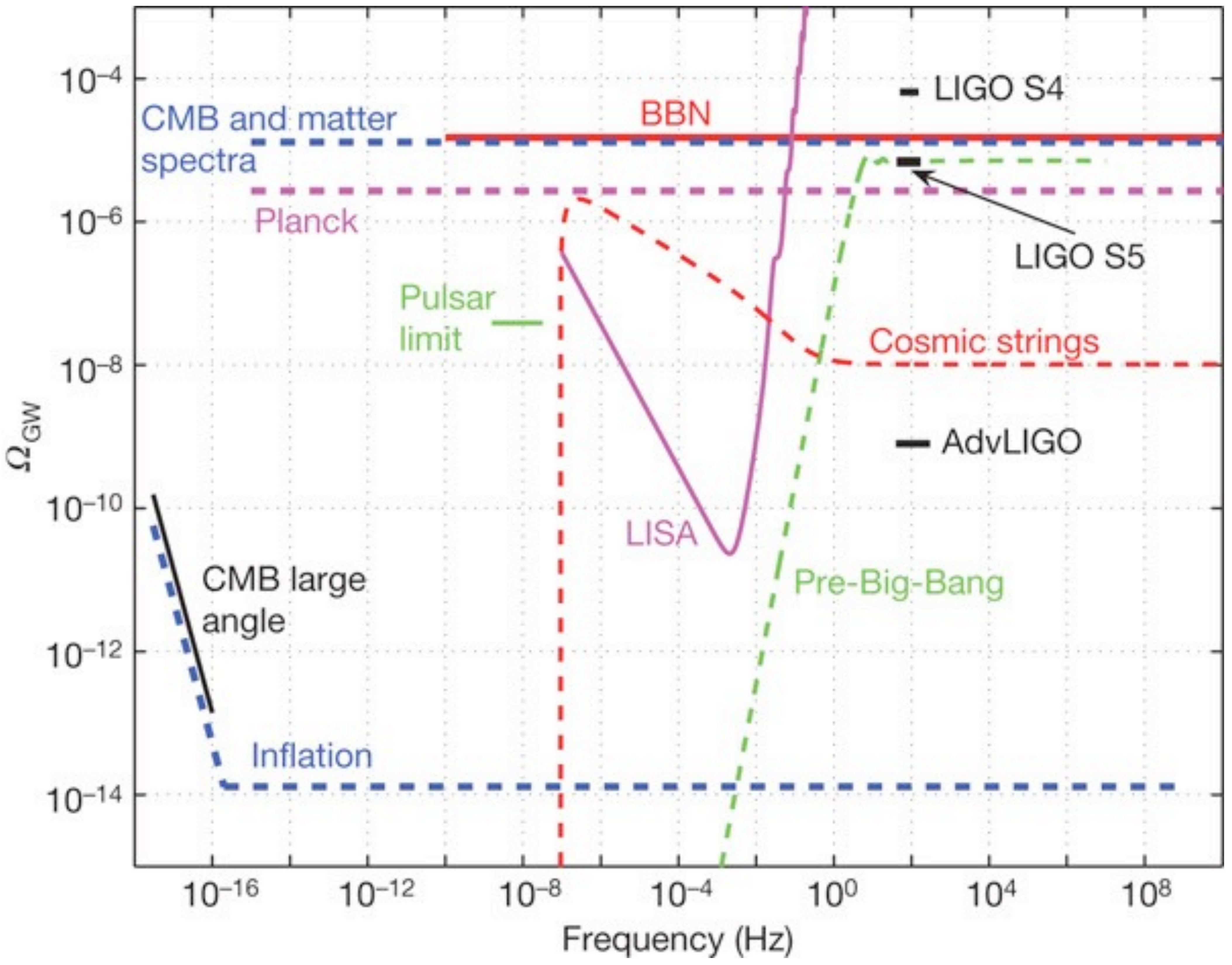


Credit: Astrobites

STOCHASTIC GW BACKGROUND (SGWB)

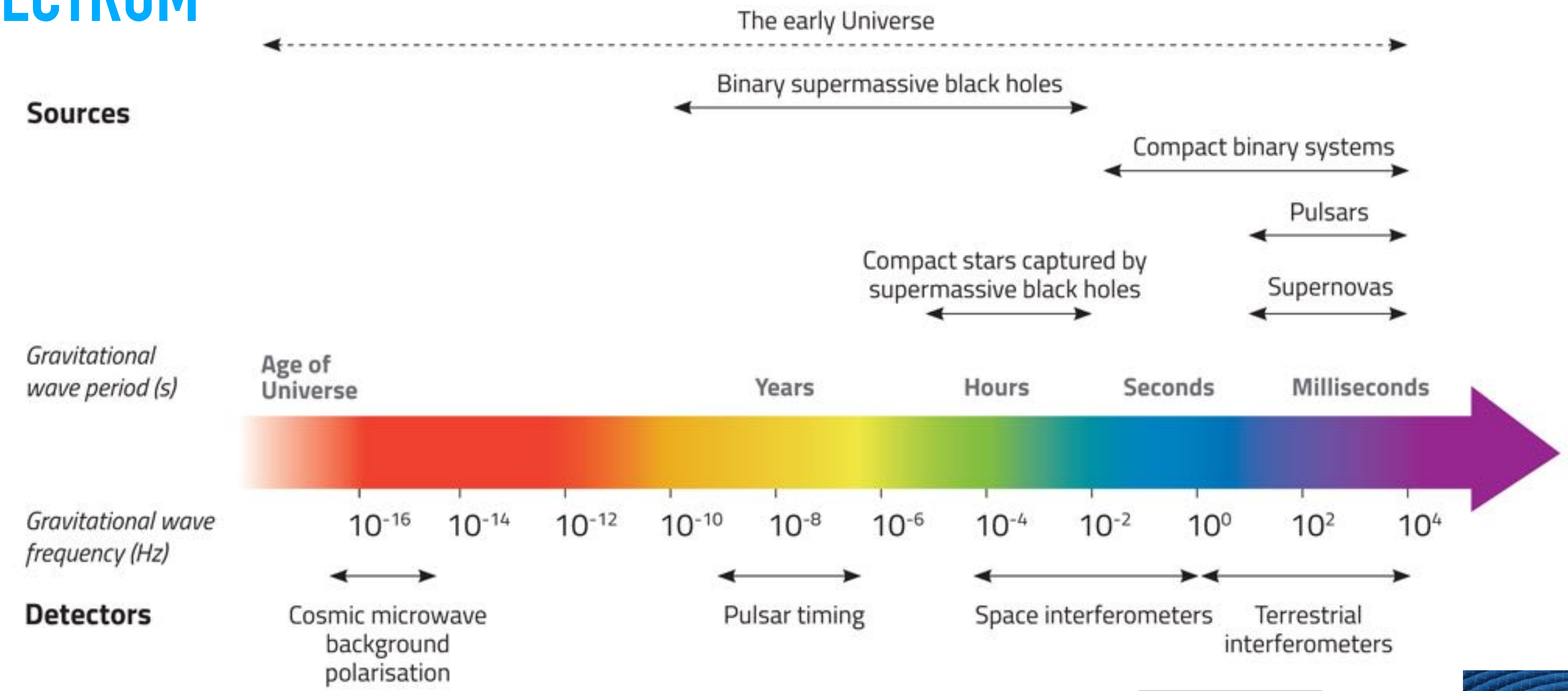
- Superposition of **astrophysical events** that cannot be resolved individually
- Background from **fundamental processes** in the early universe, e.g. the Big Bang
  - Expected to be very weak but will allow us to look back at the universe when it was $10^{-30}s$ old and at very high energies!
  - Characterised by the energy density of a random field of gravitational waves with a **mean square amplitude per unit frequency** $S_{gw}(f)$.
  - The SGWB density parameter is then given by:

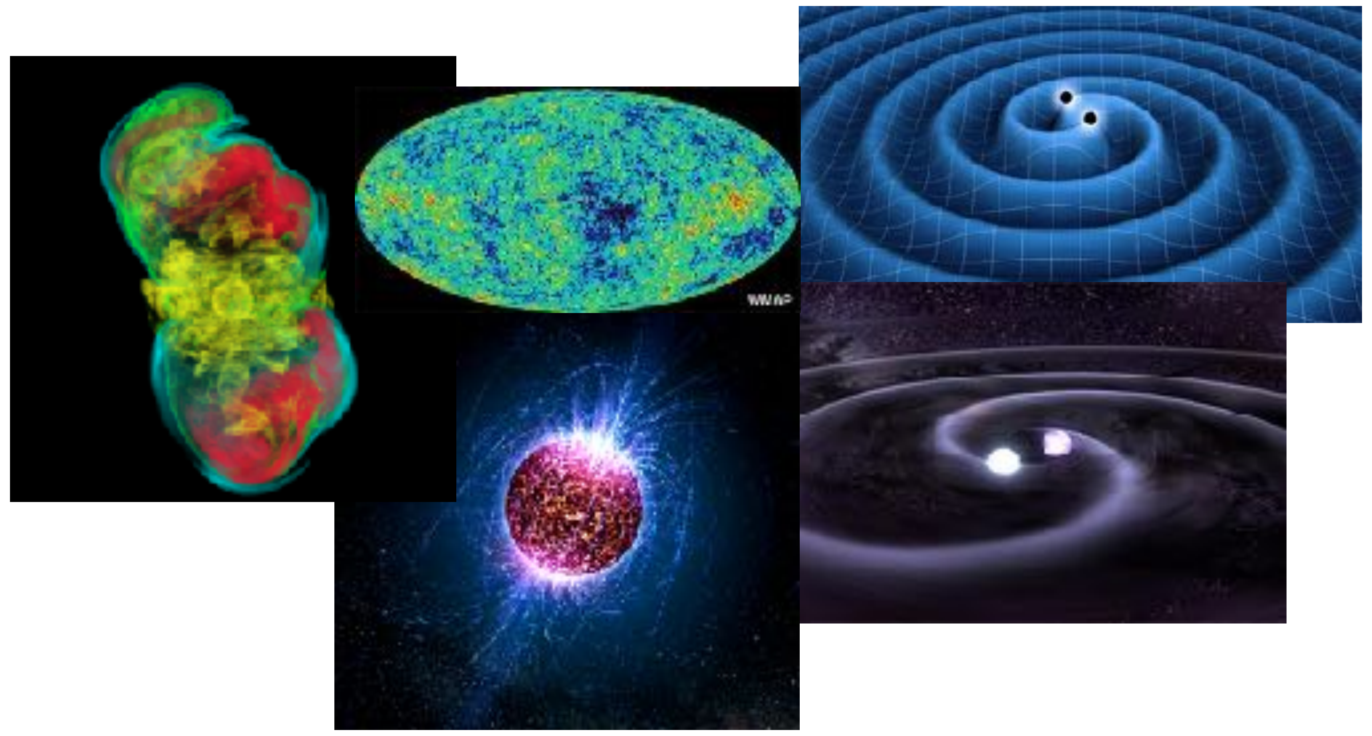
$$\Omega_{gw}(f) = \frac{10\pi^2}{3H_0^2} f^3 S_{gw}(f)$$



[LVC, Nature]

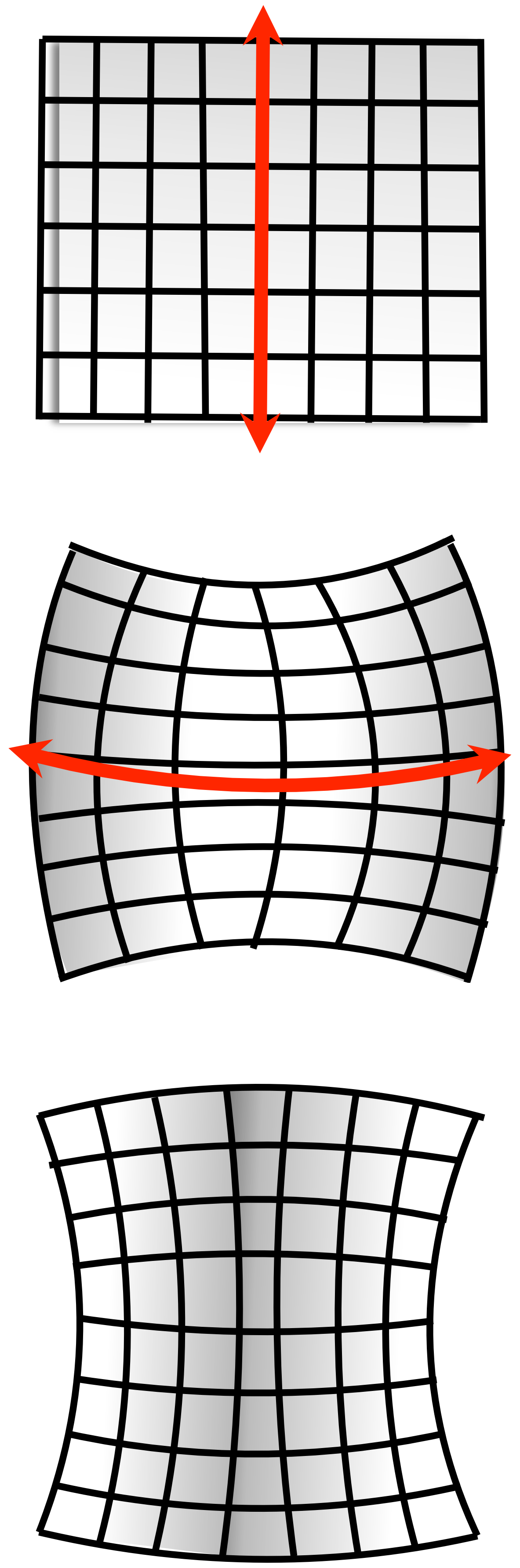
GW SPECTRUM

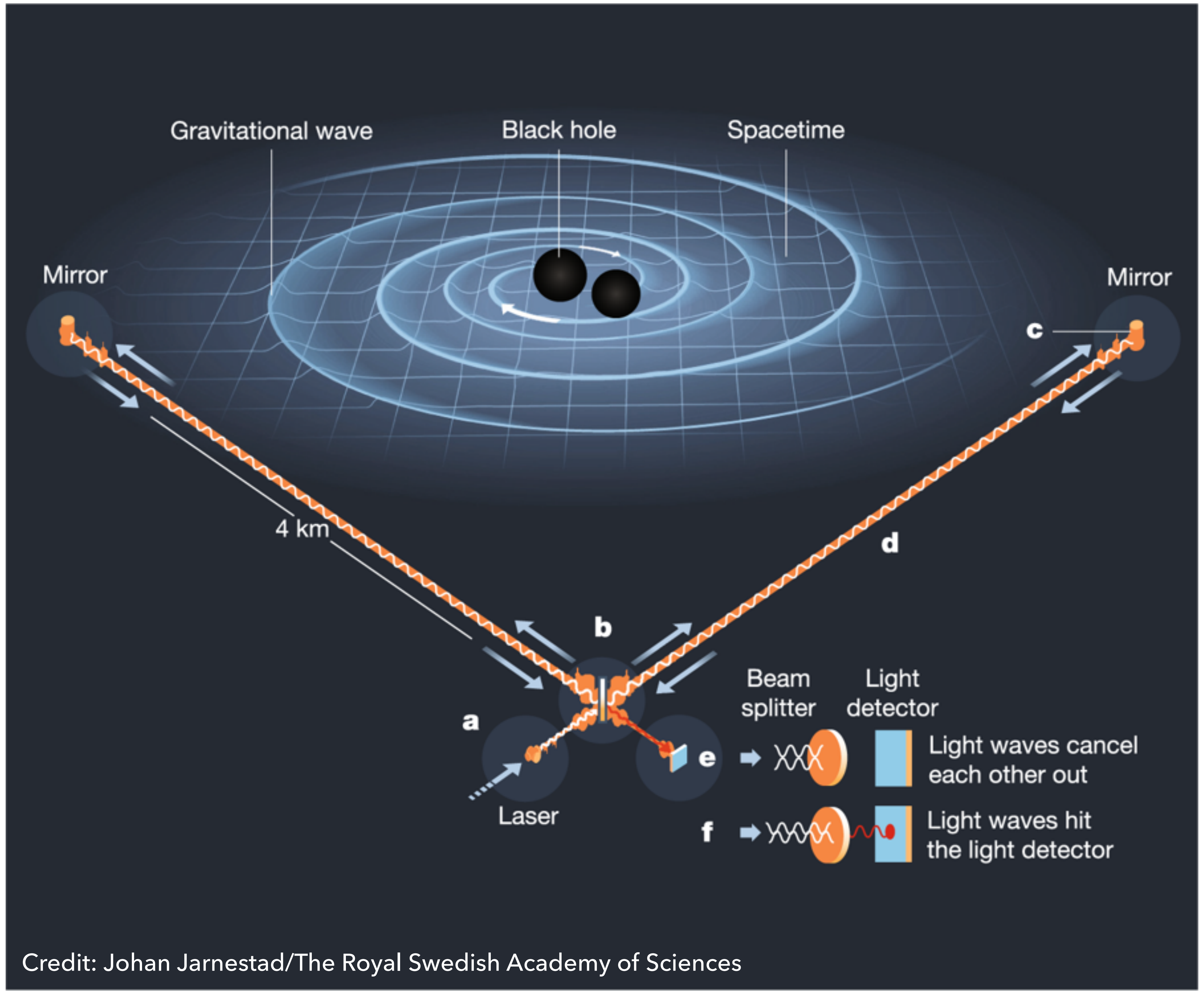

The early Universe
Sources
Binary supermassive black holes
Compact binary systems
Pulsars
Compact stars captured by supermassive black holes
Supernovas
Gravitational wave period (s)
Age of Universe
Years
Hours
Seconds
Milliseconds
Gravitational wave frequency (Hz)
10-16
10-14
10-12
10-10
10-8
10-6
10-4
10-2
100
102
104
Detectors
Cosmic microwave background polarisation
Pulsar timing
Space interferometers
Terrestrial interferometers



GRAVITATIONAL-WAVE DETECTORS

- **Precision interferometry**: Use two (perpendicular) lasers beams to measure the length of each arm



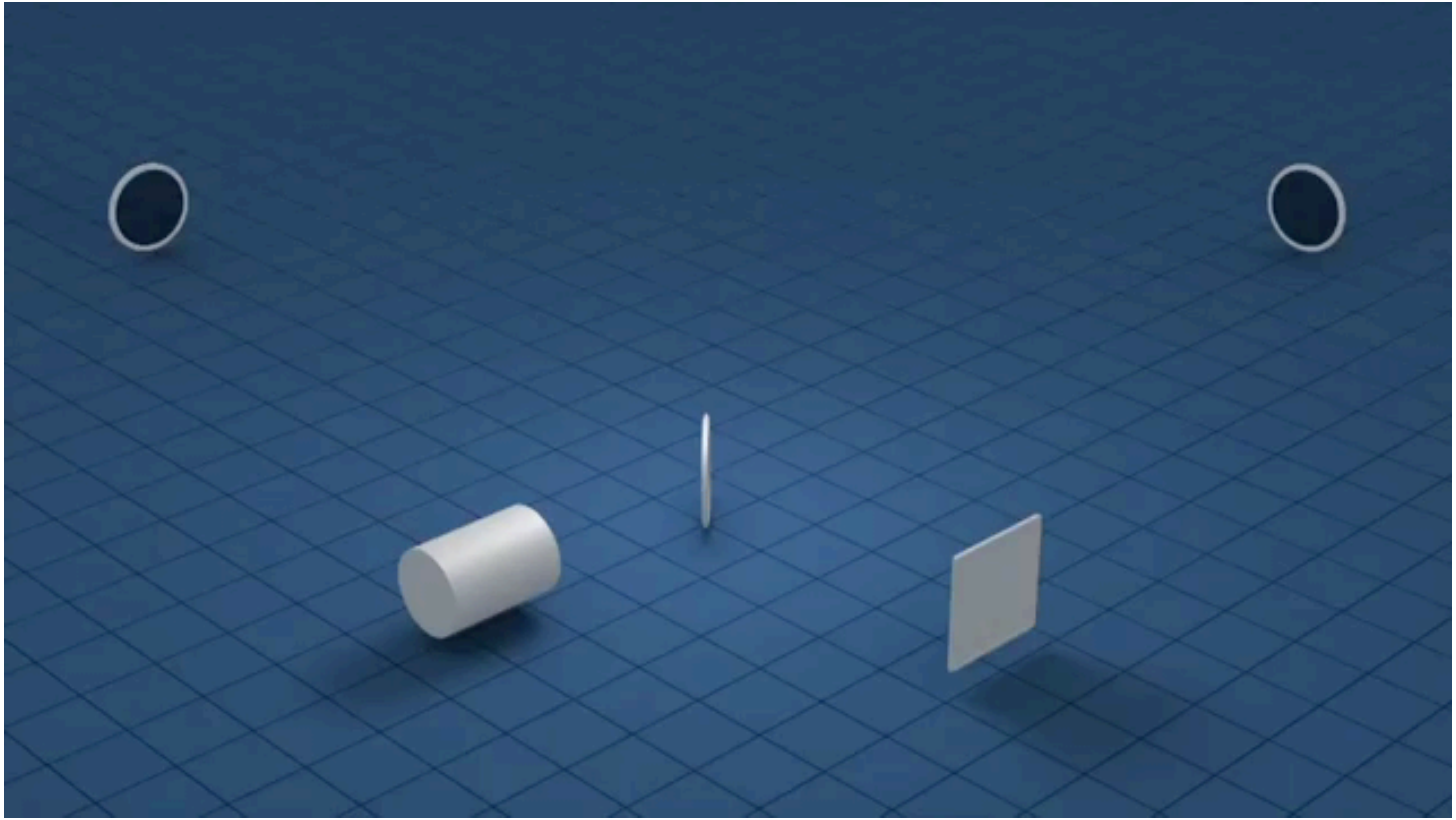


Credit: Johan Jarnestad/The Royal Swedish Academy of Sciences

Change in arm length:

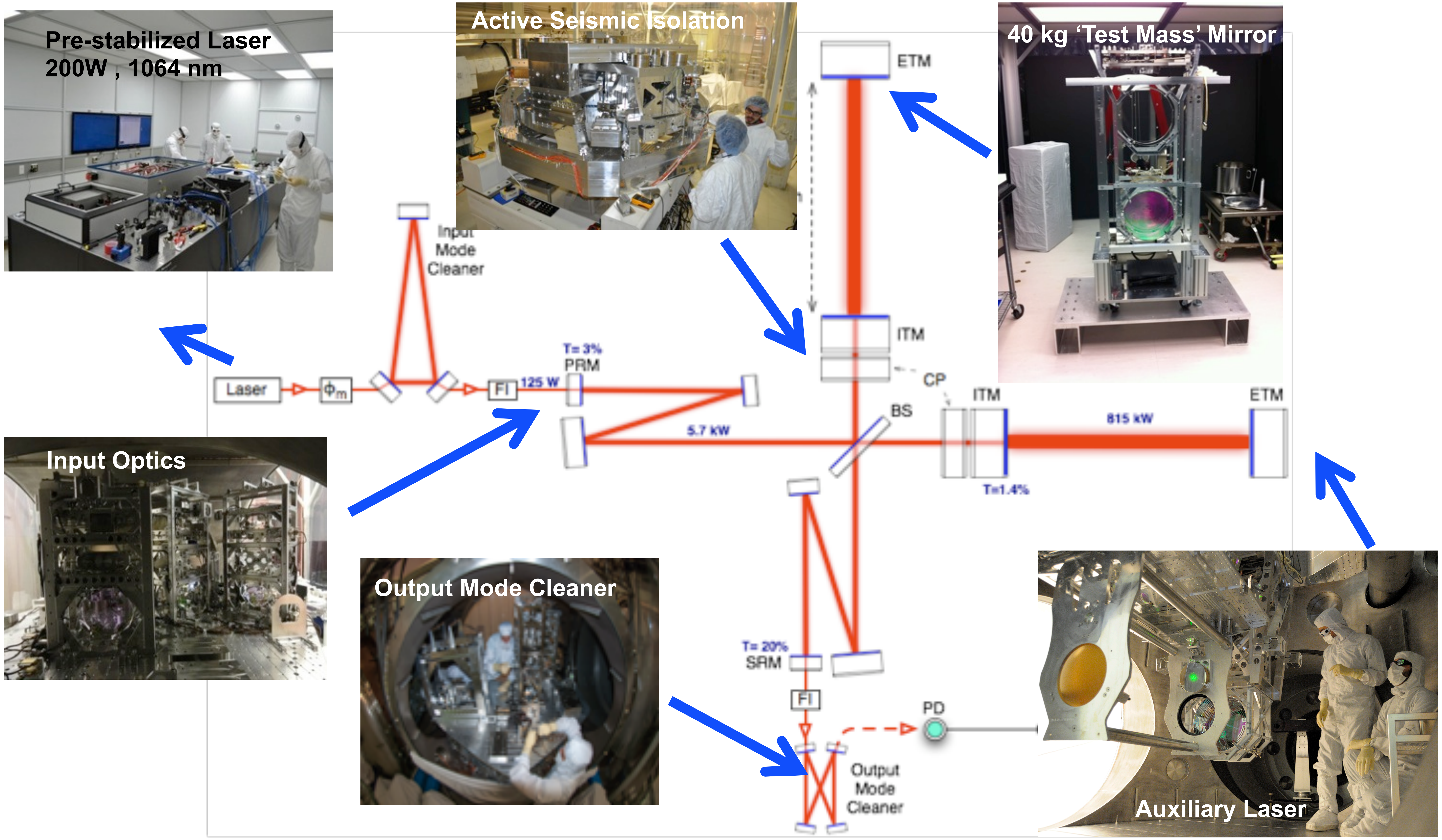
$$\Delta L \sim 10^{-18}\,\mathrm{m}$$

GRAVITATIONAL-WAVE DETECTORS



[Credit:LIGO/INFN]

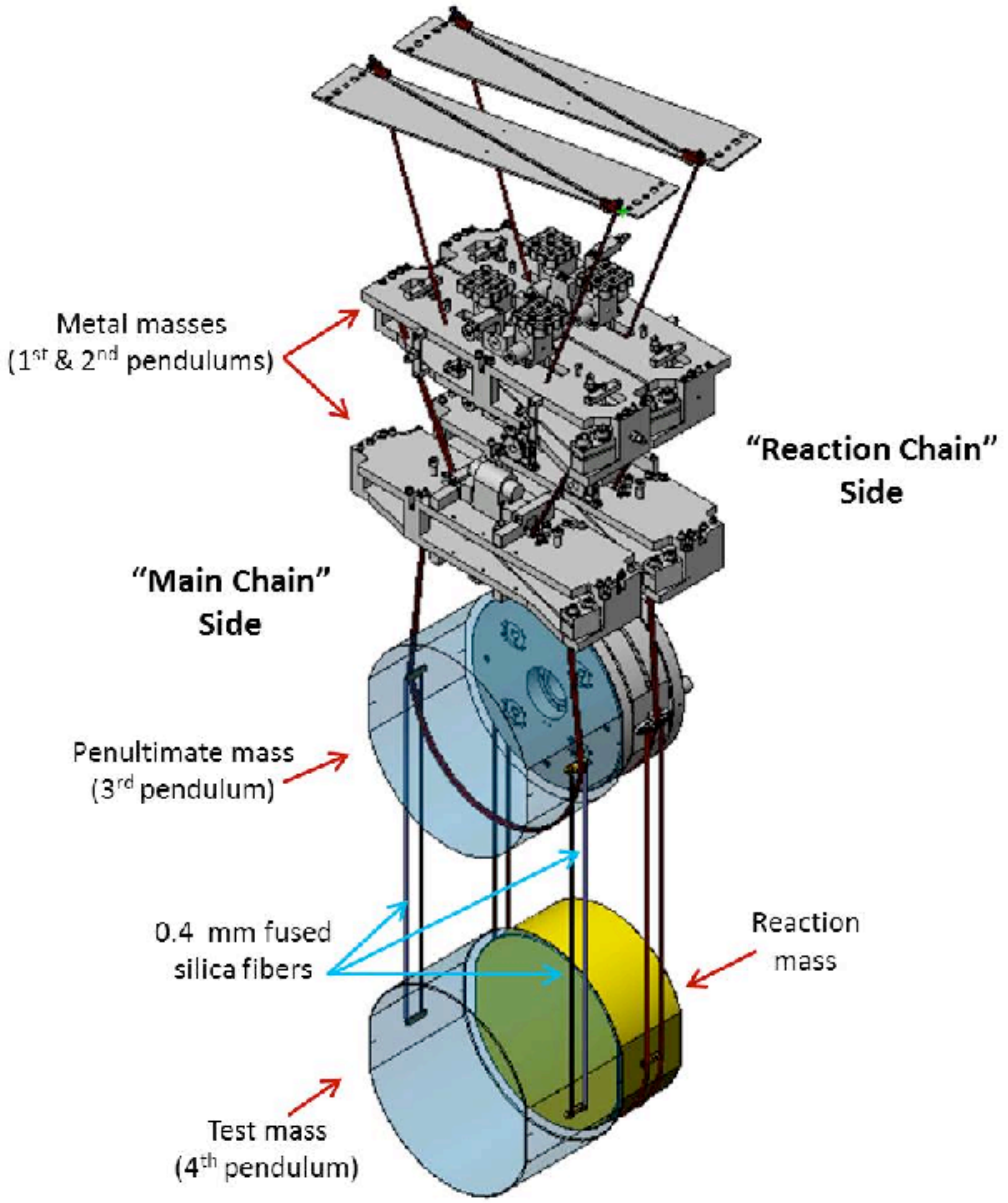
A CLOSER LOOK AT ADVANCED LIGO



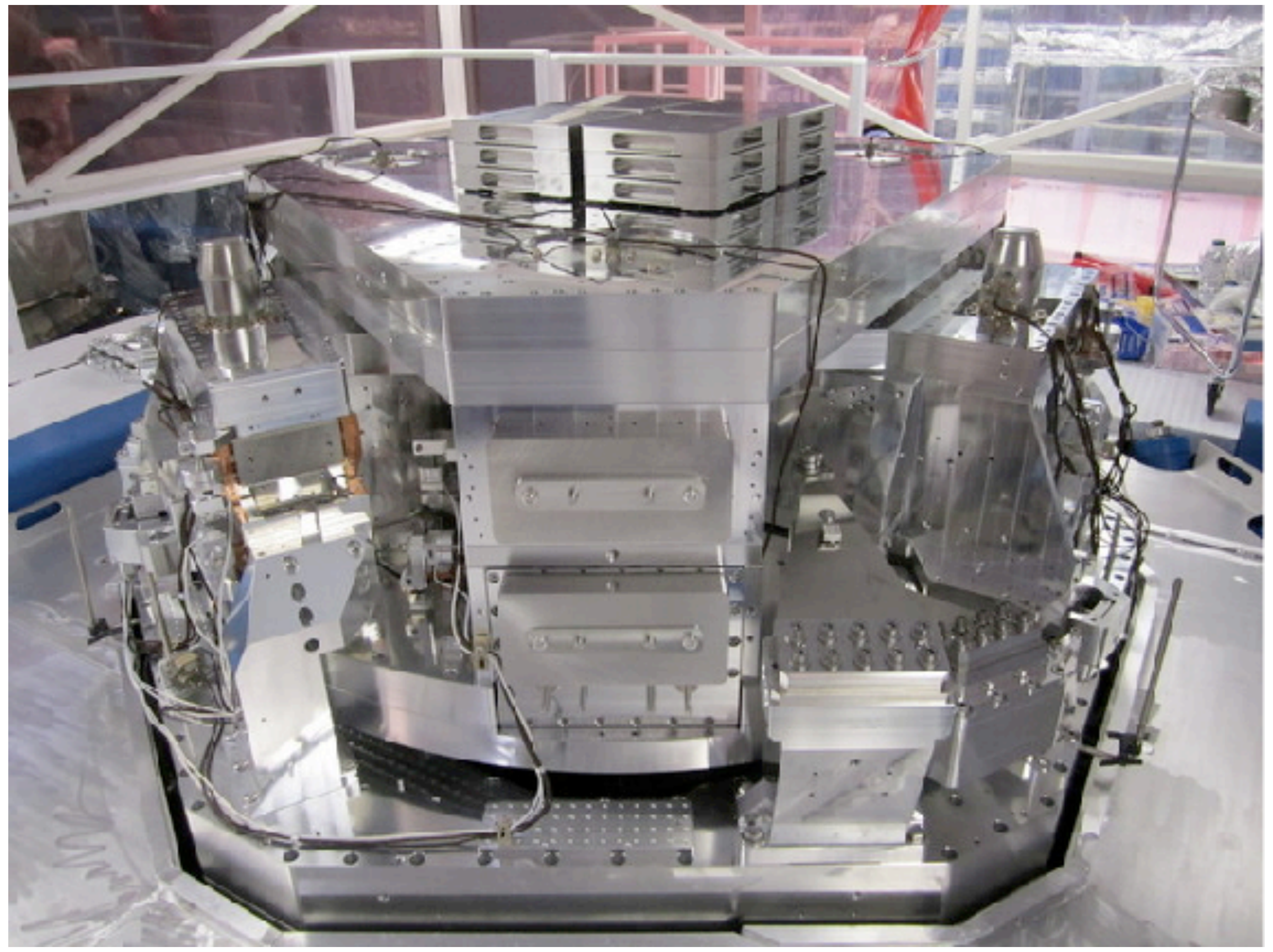
Credit: D. Reitze

A CLOSER LOOK AT ADVANCED LIGO

Multi-stage suspension system to reduce seismic noise



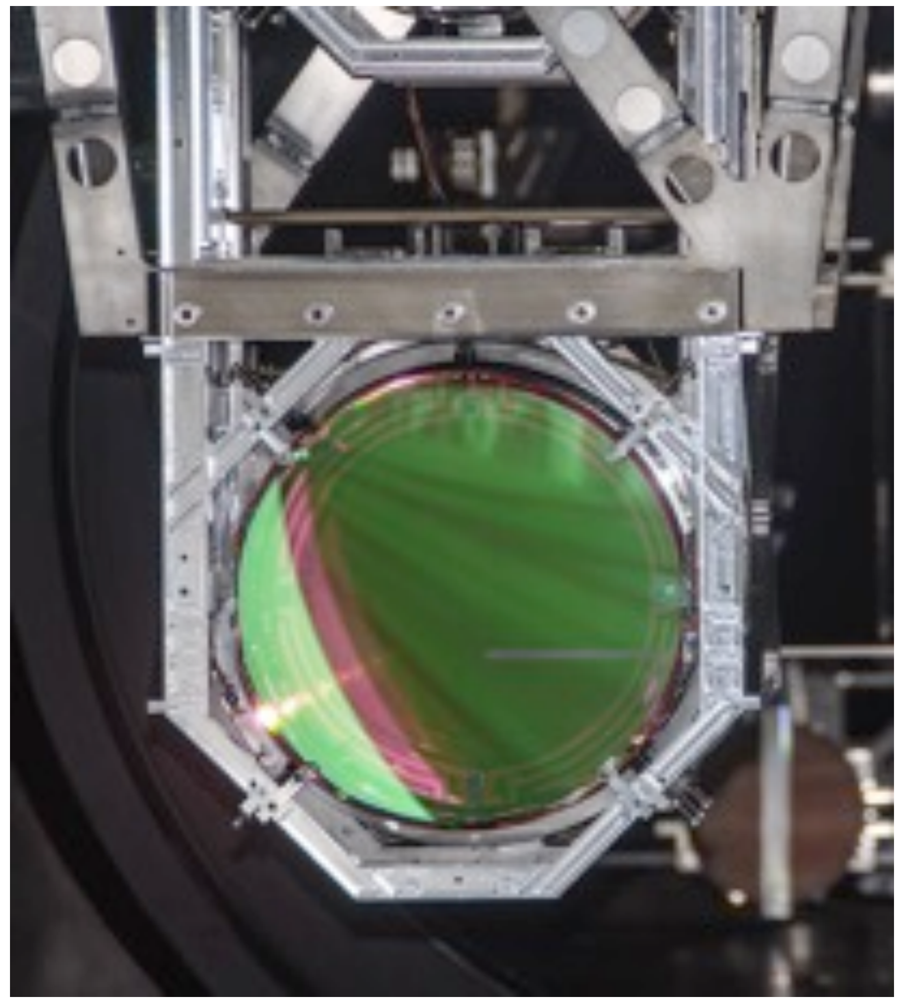
Active seismic damping platform



[Credit:LIGO]

A CLOSER LOOK AT ADVANCED LIGO

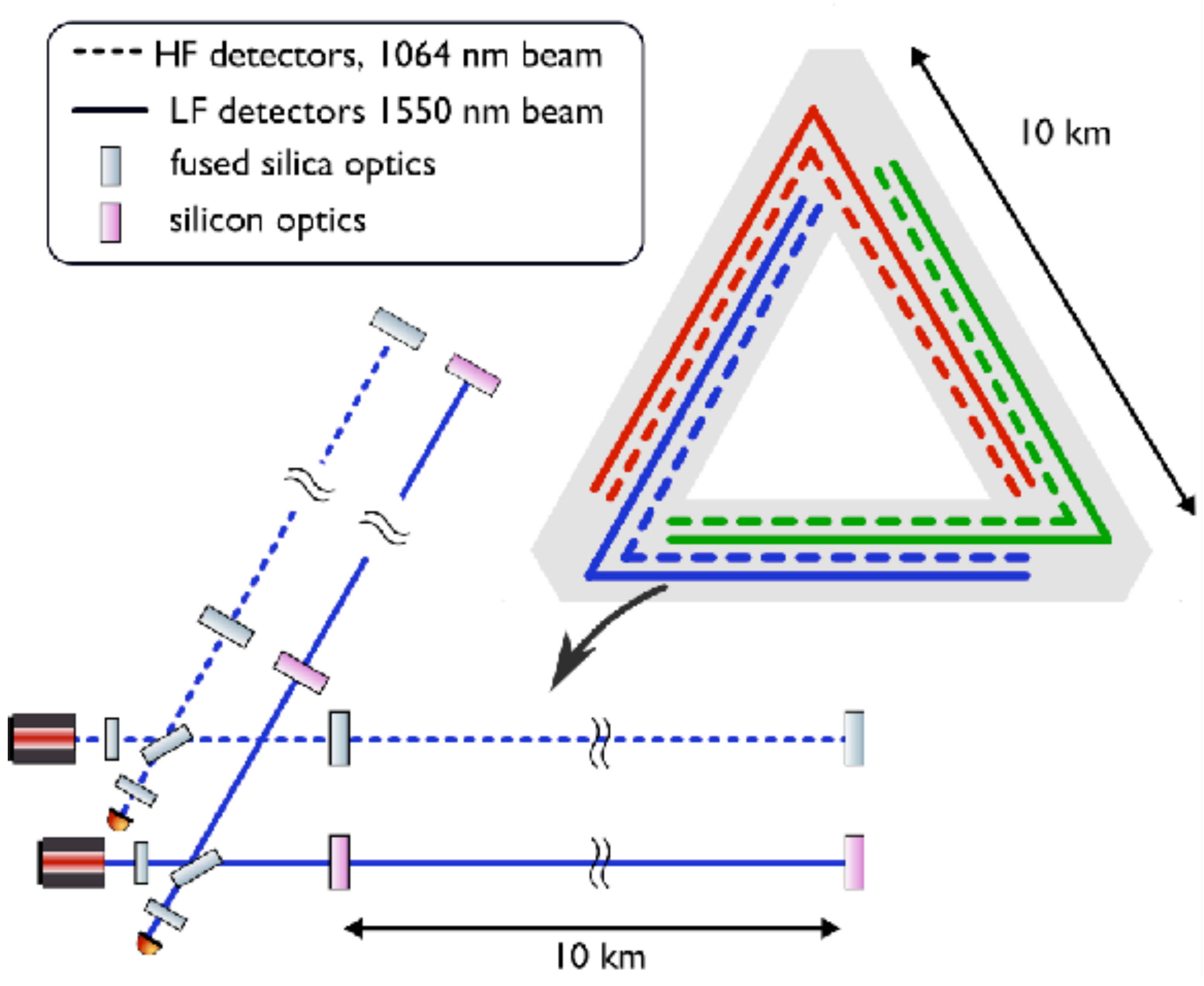
- Vacuum system for ultra-pure vacuum
  - Volume: ~9000m^3
  - Atmospheric pressure inside the tubes ~ 10^{-8}-10^{-9} Torr
  - Air molecules transfer heat onto mirrors and mimic GWs; dust can damage the mirrors
- Pre-stabilised laser + amplification
  - Input laser power in O3: 70W
  - Laser power is crucial to increase the resolution
- Mirrors: pure fused silica glass at 40kg each
  - 34 x 20 cm
  - 1-in-3-million photons get absorbed
  - Mirrors refocus the laser

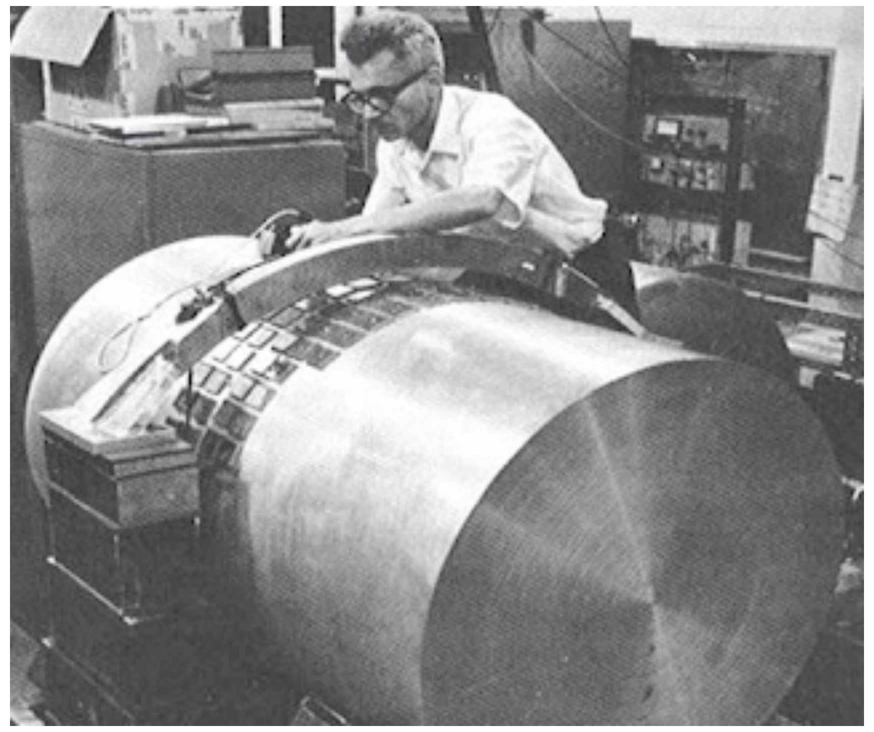


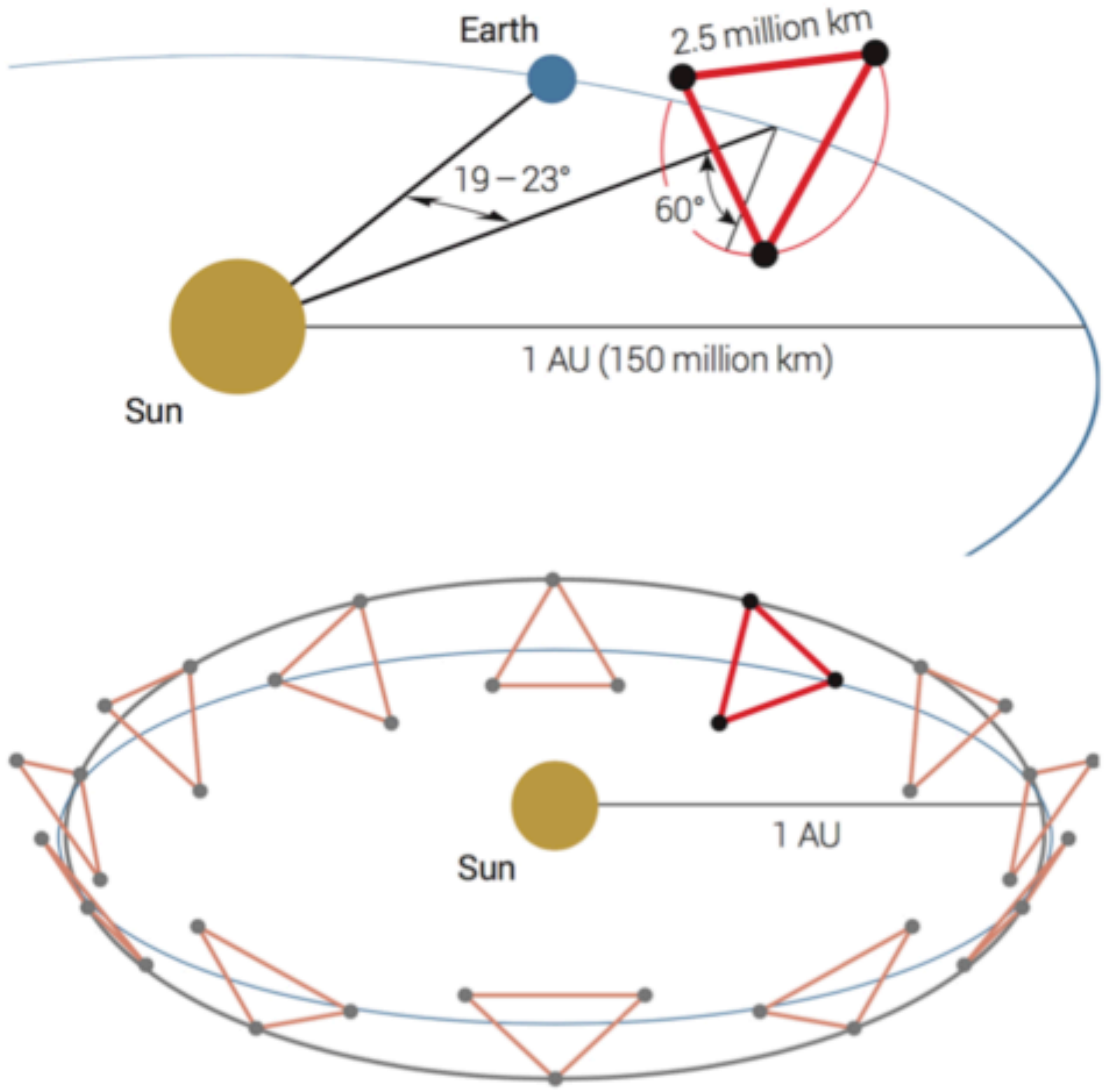
[Credit:LIGO]

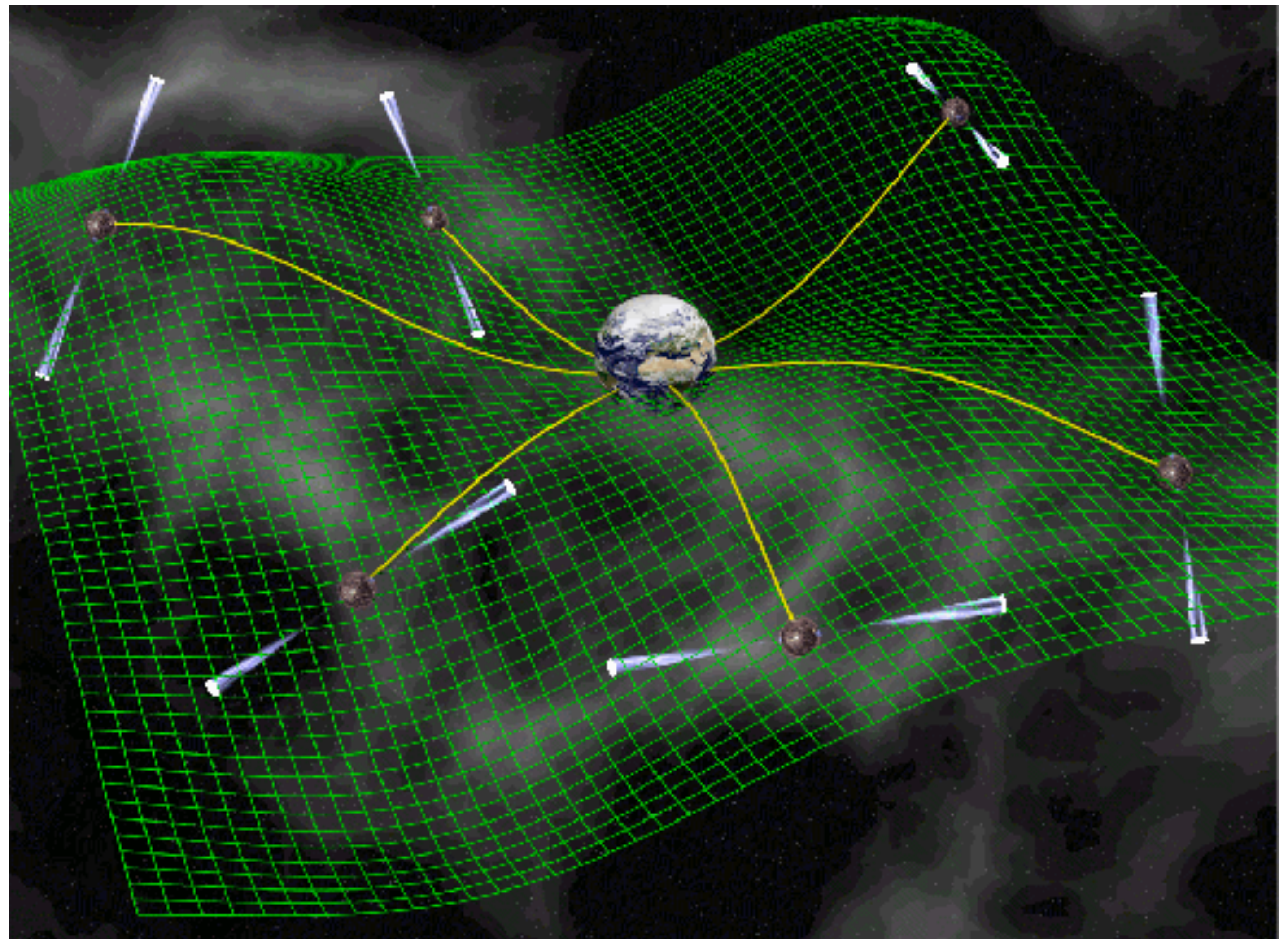
OTHER DETECTOR CONFIGURATIONS

- Triangular interferometers, e.g.
  - **Einstein Telescope (ET)**: proposed third generation ground-based detector
  - **LISA**: planned space-based mission
- Resonant bar detectors
- Pulsar timing arrays









SENSITIVITY

- The sensitivity of GW detector is characterised by the **power spectral density (PSD)** of its noise background in the absence of a GW signal.

- Data is recorded as a time series: $(n(t_0), n(t_1), \ldots, n(t_n))$

    - Discrete Fourier transform: $(\tilde{n}(f_0), \tilde{n}(f_1), \ldots, \tilde{n}(f_n))$

- Let us assume that the noise is **stationary and Gaussian**. Then the probability density of one realisation of noise per frequency bin is given by $p(\tilde{n}(f_i)) \propto e^{-|\tilde{n}(f_i)|^2/(2\sigma_i^2)}$ and total probability density for a noise realisation is $p(n) = \prod_{i=0}^{n} p(\tilde{n}(f_i))$.

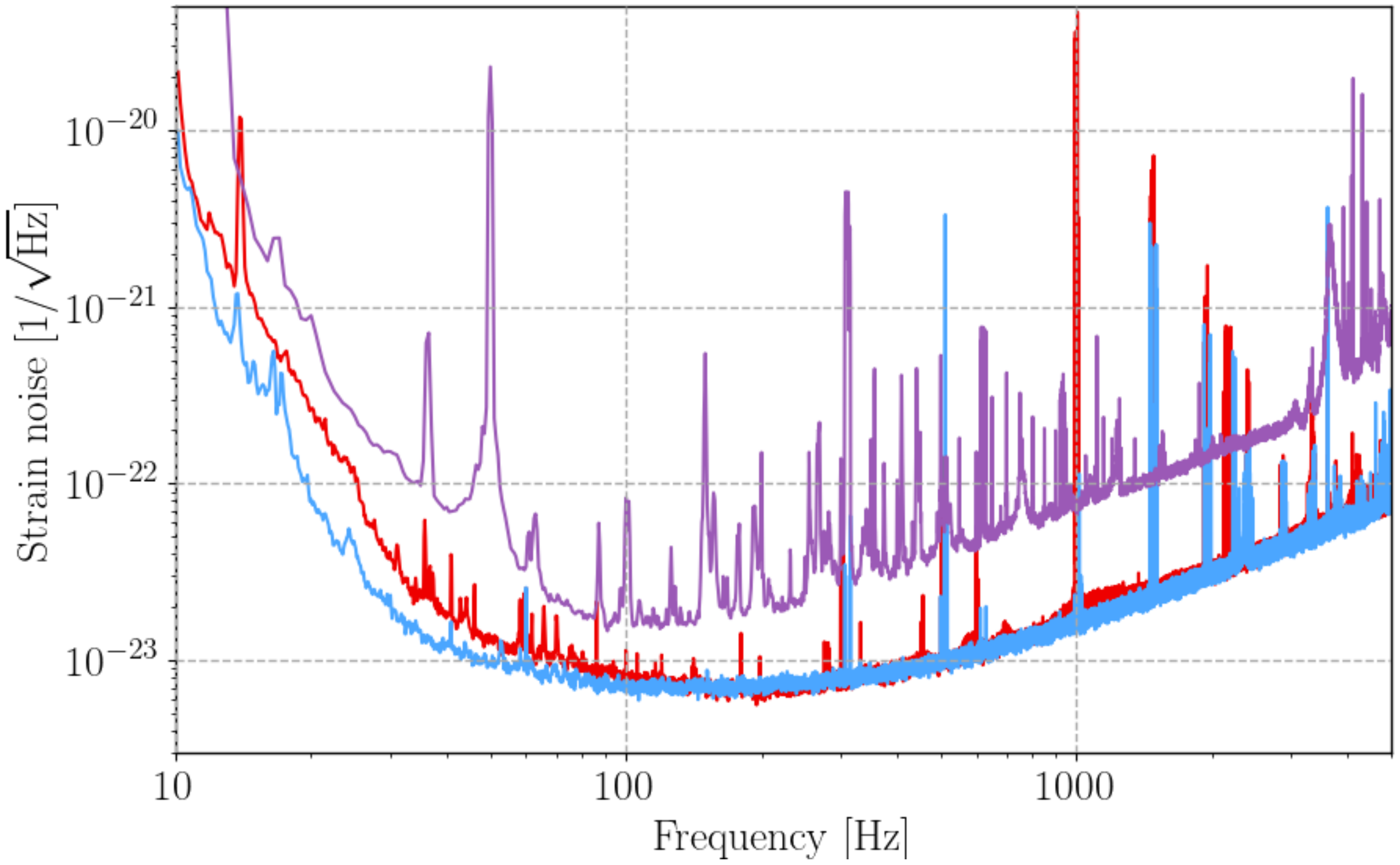
    - In the continuum limit: $p(n) = \mathcal{N} e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{|\tilde{n}(f_i)|^2}{\sigma_i^2}} \rightarrow \mathcal{N} e^{-\int_{-\infty}^{\infty} \frac{|\tilde{n}(f)|^2}{S_n(f)} df}$

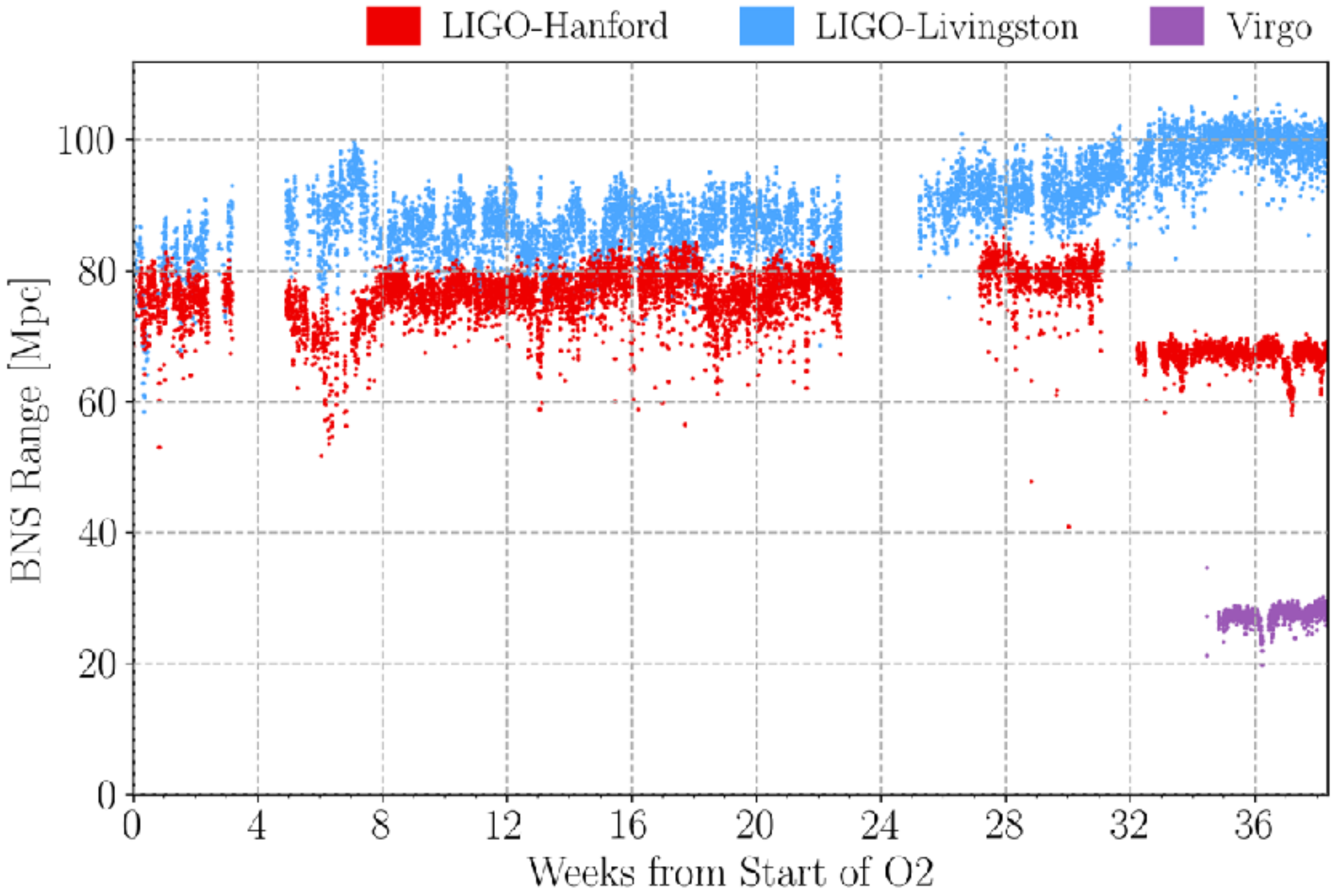
    - $S_n(f)$ is the **noise power spectral density** - the Fourier transform of the noise autocorrelation function:

$$\langle \tilde{n}(f)\tilde{n}^*(f') \rangle = \frac{1}{2} S_n(f)\delta(f - f')$$

SENSITIVITY DURING O1/O2

Amplitude spectral density = $\sqrt{\mathrm{PSD}}$

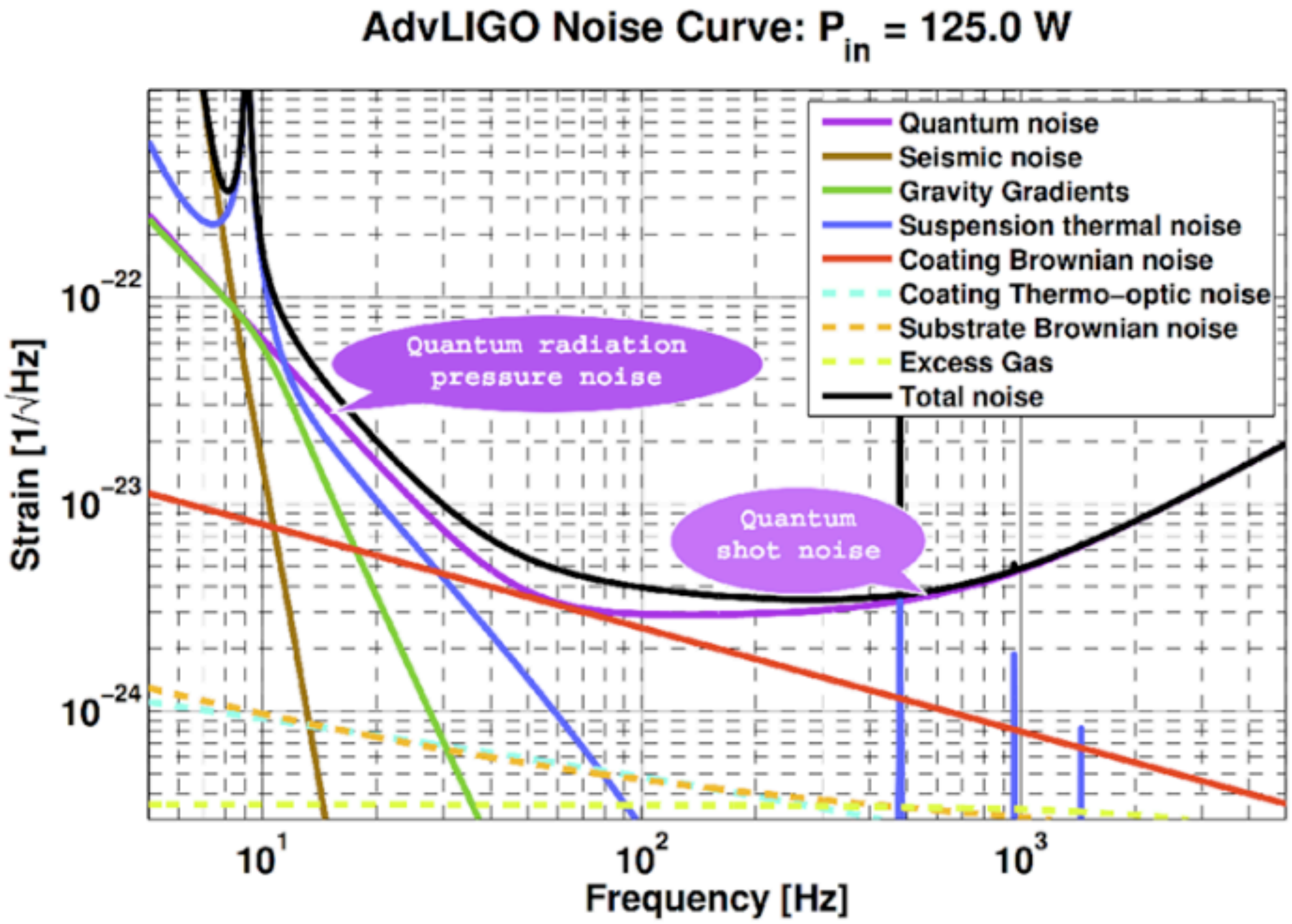




Range: Sky and orientation averaged distance such that a BNS has a SNR of 8

[LVC, GWTC-1]

MAJOR SOURCES OF NOISE



[Living Rev. Rel. 2019 22:2]