Cosmic ray theory (II)

Phillip Mertsch

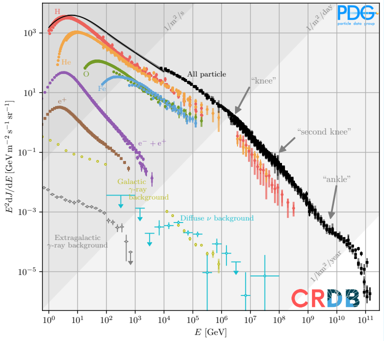
International School of Cosmic Ray Astrophysics
30 July 2026

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY

Outline

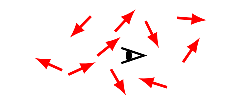
Recap (I): Fundamental observations

Spectrum



- Power law $\rightarrow$ stochastic acceleration
- Features $\rightarrow$ origin

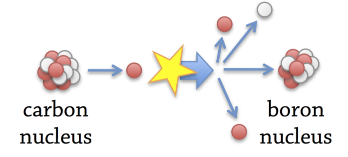
Anisotropies



- Dipole $a = \mathcal{O}(10^{-3\ldots-2})$

$\rightarrow$ Requires isotropisation

Composition

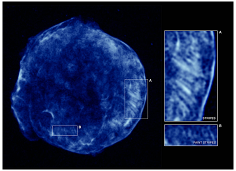


- Primaries: present in sources
- Secondaries: produced on route

$\rightarrow$ Requires isotropisation

Recap (II): Key insights

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{N_1}{t_{\mathrm{esc}}} - N_1\Gamma_1 + Q_1$$
$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -\frac{N_2}{t_{\mathrm{esc}}} - N_2\Gamma_2 + \Gamma_{1\to 2}N_1$$



The slab model

- Definition of grammage
- From B/C: average grammage of $\sim$ (a few) $\mathrm{g\,cm^{-2}}$
- Requires crossing the disk thousands of times

The leaxy-box model

- $t_{\mathrm{res}} = \mathcal{O}(10)\,\mathrm{Myr}$
- Cosmic-ray clocks
- Grammage rigidity-dependent

Supernova remnants

1. Presence of strong shocks
2. Observation of PeV particles
3. Energetics

Outline

Definitions

- We will consider the differential (in momentum p) particle density ψ,

$$\psi(\mathbf{r}, p, t) = 4\pi p^2 f_0(\mathbf{r}, p, t) = \frac{\mathrm{d}n}{\mathrm{d}p}$$

- Can also express this as differential in energy E, $\tilde{\psi}$,

$$\tilde{\psi}(\mathbf{r}, p, t) = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{\mathrm{d}p}{\mathrm{d}E}\psi(\mathbf{r}, p, t) = \frac{E}{pc^2}\psi(\mathbf{r}, p, t)$$

- What is measured is the intensity

$$J(\mathbf{r}, p, t) = \frac{\mathrm{d}^4 N}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}E} = \frac{v}{4\pi}\tilde{\psi}(\mathbf{r}, p, t) = p^2 f_0(\mathbf{r}, p, t)$$

The transport equation

$$\frac{\partial \psi_j}{\partial t} = \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j)$$

spatial diffusion and advection

$$+ \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right)$$

momentum diffusion

$$+ \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} (\nabla \cdot \mathbf{U}) \psi_j \right)$$

momentum change incl. adiabatic

$$- v n_{\mathrm{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j}$$

spallation and decay

$$+ v n_{\mathrm{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}}$$

spallation and decay

$$+ S_j$$

primary sources

Mathematical description

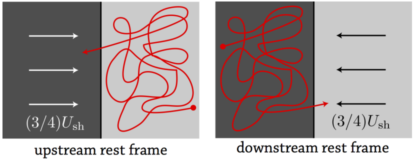
Ginzburg & Syrovatskii (1964)

Transport equation

$$\frac{\partial \psi_j}{\partial t} - \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{u}\psi_j) + \frac{\partial}{\partial p}\left(\frac{p}{3}(\nabla \cdot \mathbf{u})\,\psi_j\right) = q_j$$

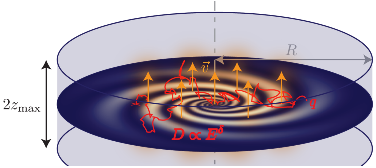
↓

Application to blast wave:



↓

Application to galactic halo:



Shock acceleration

Source spectrum: $q(\mathcal{R}) \propto \mathcal{R}^{-2.4\ldots-1.9}$

Axford, Leer, Skadron (1977); Krymskii (1977); Bell (1978); Blandford, Ostriker (1978)

Diffusive escape

Observed spectrum: $\psi(\mathcal{R}) \propto \frac{q(\mathcal{R})}{\kappa(\mathcal{R})}$ e.g. $\frac{\mathcal{R}^{-2.2}}{\mathcal{R}^{0.6}} \propto \mathcal{R}^{-2.8}$

The transport equation

$$
\begin{aligned}
\frac{\partial \psi_j}{\partial t} = & \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j) && \text{spatial diffusion and advection} \\
& + \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right) && \text{momentum diffusion} \\
& + \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} \left(\nabla \cdot \mathbf{U} \right) \psi_j \right) && \text{momentum change incl. adiabatic} \\
& - v n_{\mathrm{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j} && \text{spallation and decay} \\
& + v n_{\mathrm{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}} && \text{spallation and decay} \\
& + S_j && \text{primary sources}
\end{aligned}
$$

CRs are a collisionless plasma

- From local measurement $\varepsilon_{\mathrm{CR}} \sim 1\,\mathrm{eV\,cm}^{-3} \Rightarrow n_{\mathrm{CR}} \sim 10^{-9}\mathrm{cm}^{-3}$
- Inelastic collisions, typically $\sigma \sim 100\,\mathrm{mb} = 10^{-25}\,\mathrm{cm}^2$

$$\begin{aligned} \Rightarrow \lambda &= (n_{\mathrm{CR}}\sigma)^{-1} \\ &= 10^{34}\,\mathrm{cm} \simeq 3 \times 10^{12}\,\mathrm{kpc} \end{aligned}$$

$\rightarrow$ (mean free path) $\gg$ (size of the system)

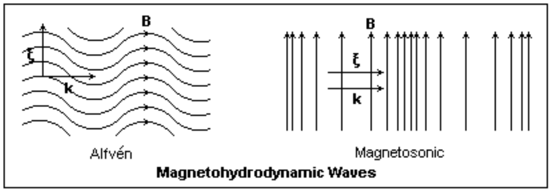
$\rightarrow$ CRs described by the collisionless Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f = 0$$

Diffusion

- But CRs interact with the magnetic field in the background plasma
- Homogeneous field $\mathbf{B}_0 \to$ gyration
- Magneto-hydrodynamic (MHD) equations allow for small-amplitude solutions $\delta\mathbf{B}$: plasma waves

Examples

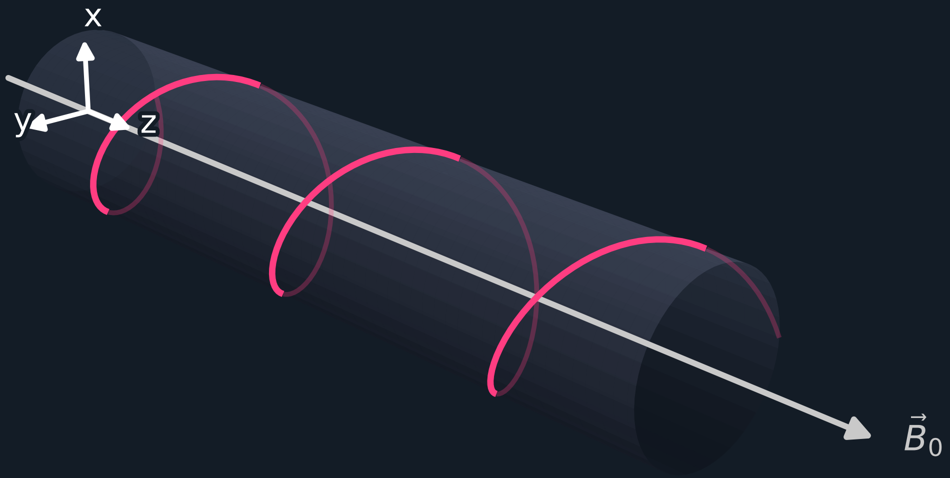


- Can perturbatively derive a diffusion equation:

$$\frac{\partial \psi}{\partial t} - \nabla \cdot (\kappa \cdot \nabla - \mathbf{U})\, \psi = 0$$

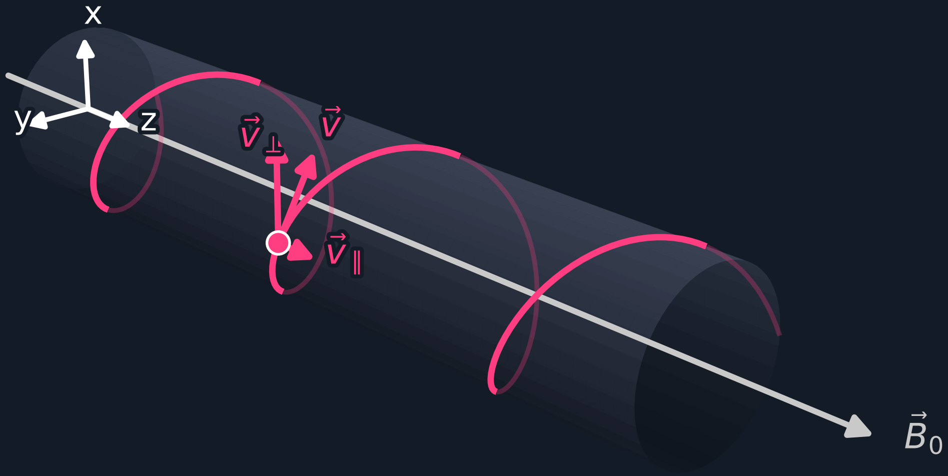
Resonance condition

Resonance condition



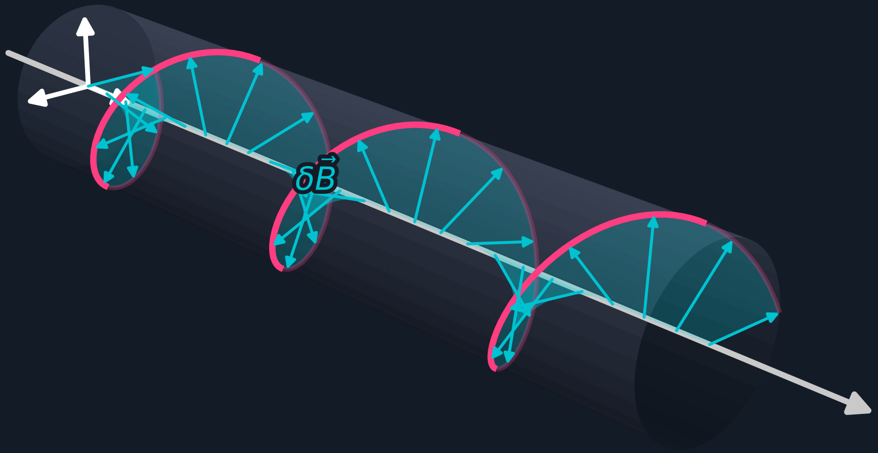
Cosmic ray particle

Resonance condition



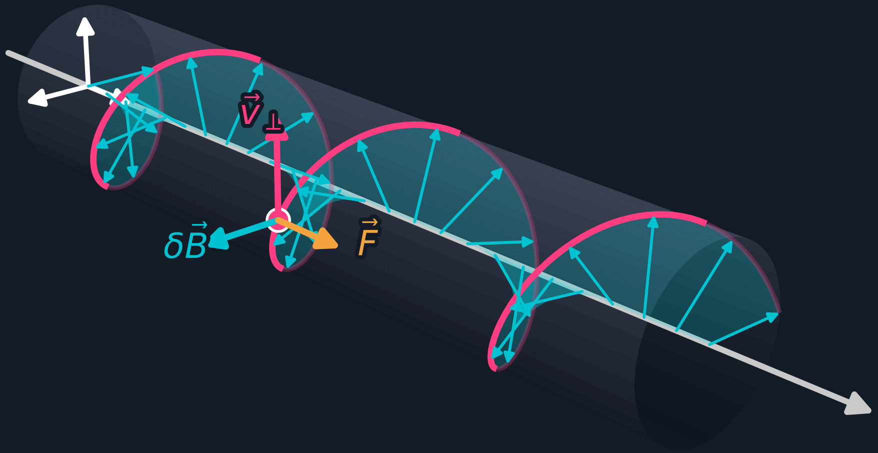
Cosmic ray particle

Resonance condition



Cosmic ray particle + Circularly polarised plasma wave (in phase)

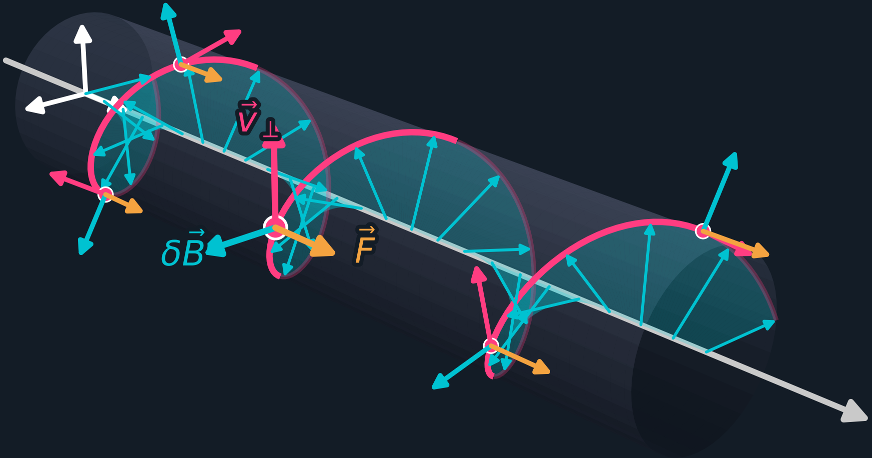
Resonance condition



Cosmic ray particle + Circularly polarised plasma wave (in phase)

Lorentz force: $\mathbf{F} = q\dfrac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

Resonance condition

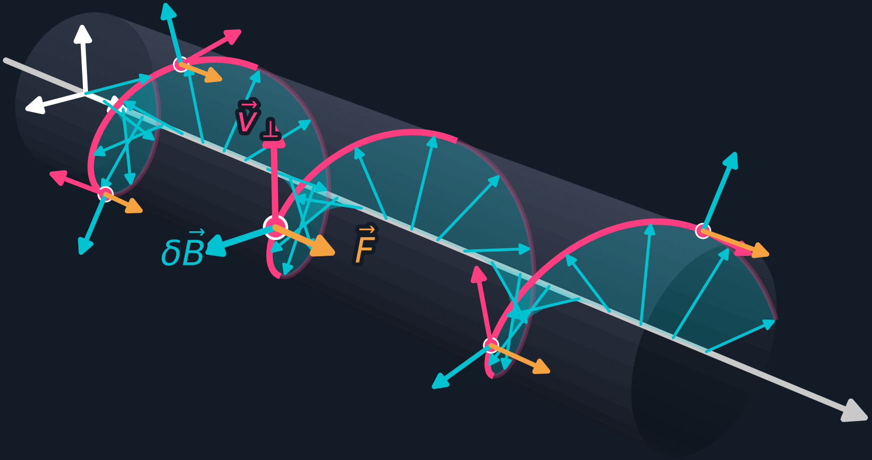


Cosmic ray particle + Circularly polarised plasma wave (in phase)

Lorentz force: $\mathbf{F} = q\frac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

If Cosmic ray particle and wave in phase $\Rightarrow$ coherent force

Resonance condition



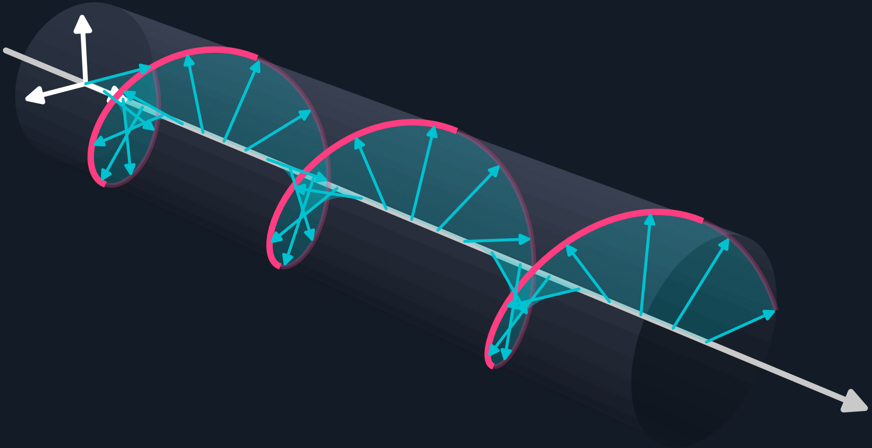
Cosmic ray particle + Circularly polarised plasma wave (in phase)

Lorentz force: $\mathbf{F} = q \dfrac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

If Cosmic ray particle and wave in phase $\Rightarrow$ coherent force

$$k\, v_\parallel = \pm\Omega$$

Resonance condition

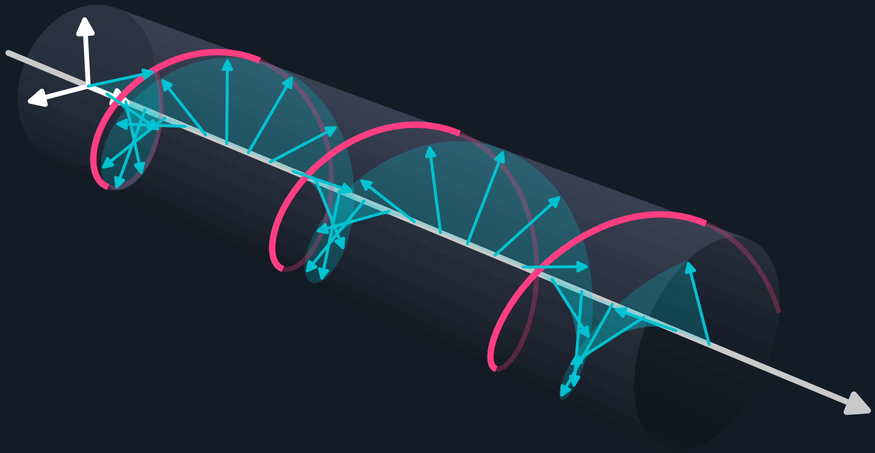


Cosmic ray particle + Circularly polarised plasma wave (in phase)

Lorentz force: $\mathbf{F} = q \frac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

If Cosmic ray particle and wave in phase $\Rightarrow$ coherent force

Resonance condition

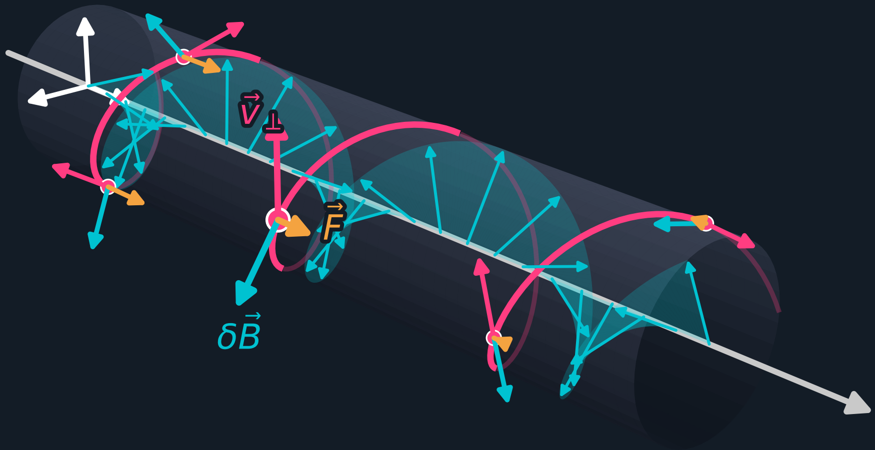


Cosmic ray particle + Circularly polarised plasma wave (**NOT** in phase)

Lorentz force: $\mathbf{F} = q \frac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

If Cosmic ray particle and wave **NOT** in phase $\Rightarrow$ **NO** coherent force

Resonance condition

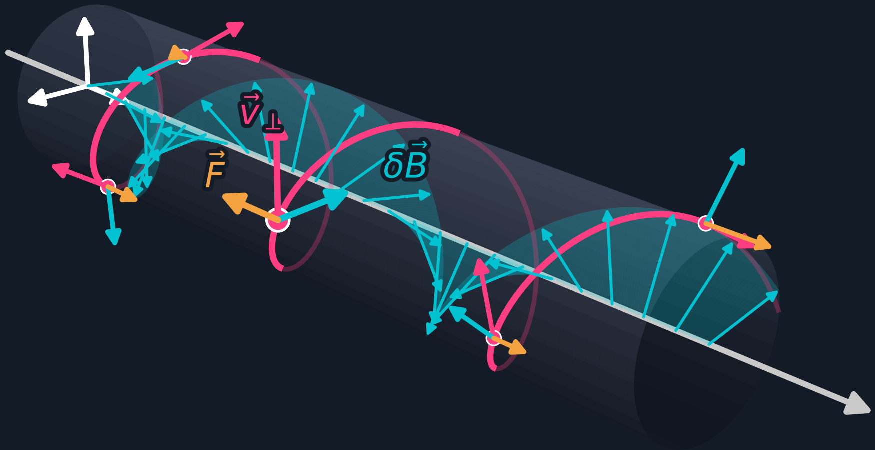


Cosmic ray particle + Circularly polarised plasma wave (**NOT** in phase)

Lorentz force: $\mathbf{F} = q \frac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

If Cosmic ray particle and wave **NOT** in phase ⇒ **NO** coherent force

Resonance condition



Cosmic ray particle + Circularly polarised plasma wave (**NOT** in phase)

Lorentz force: $\mathbf{F} = q\dfrac{\mathbf{v}_\perp \times \delta\mathbf{B}}{c}$

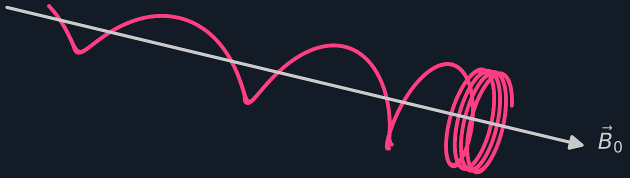
If Cosmic ray particle and wave **NOT** in phase $\Rightarrow$ **NO** coherent force

Pitch-angle scattering

Resonance condition $k\, v_{||} = \pm\Omega$ satisfied
$\Rightarrow$ pitch-angle $\sphericalangle\,(\mathbf{v}, \mathbf{B}_0)$ changes

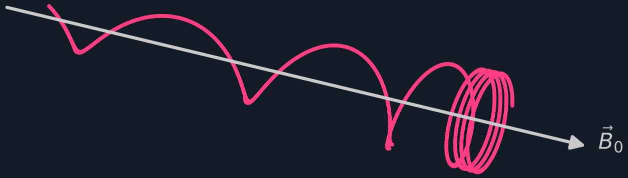
Pitch-angle scattering

Resonance condition $k\,v_{\|} = \pm\Omega$ satisfied
$\Rightarrow$ pitch-angle $\sphericalangle\,(\mathbf{v}, \mathbf{B}_0)$ changes

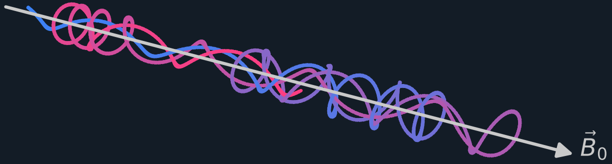


Pitch-angle scattering

Resonance condition $k\, v_{\|} = \pm\Omega$ satisfied
$\Rightarrow$ pitch-angle $\sphericalangle\,(\mathbf{v}, \mathbf{B}_0)$ changes



Interacting with waves with random phases
$\Rightarrow$ Random walk of pitch-angle
$\Rightarrow$ Pitch-angle scattering

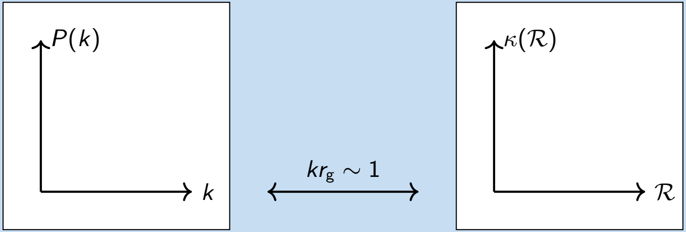


Diffusion

- Resonace condition: $$k = \frac{\Omega}{v} \propto p^{-1}$$
- Scattering rate $$\Gamma(k) \simeq \Omega \frac{\delta \tilde{B}^2(k)}{B_0^2} \simeq \Omega \frac{k\, P(k)}{B_0^2}$$
- Wave power spectrum $$P(k) \sim \int \mathrm{d}^3 r \, \mathrm{e}^{i\mathbf{r}\cdot\mathbf{k}} \langle \delta \mathbf{B}(\mathbf{r}_0) \delta \mathbf{B}(\mathbf{r}_0 + \mathbf{r}) \rangle \propto k^{-5/3}$$
- Pitch-angle scattering induces spatial diffusion:

$$\kappa \simeq \frac{c^2}{3\Gamma} \propto \frac{1}{\Omega} \left(\frac{k\, P(k)}{B_0^2} \right)^{-1} \propto p^{1/3}$$

Resonant interactions

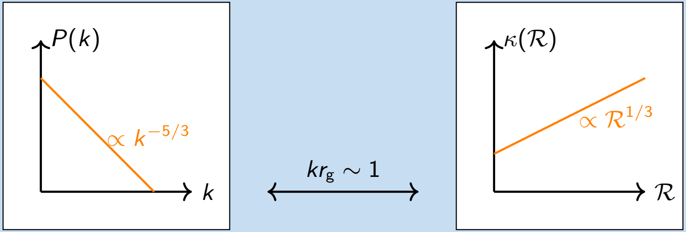


Diffusion

- Resonace condition: $$k = \frac{\Omega}{v} \propto p^{-1}$$
- Scattering rate $$\Gamma(k) \simeq \Omega \frac{\delta \tilde{B}^2(k)}{B_0^2} \simeq \Omega \frac{k\, P(k)}{B_0^2}$$
- Wave power spectrum $$P(k) \sim \int \mathrm{d}^3 r\, \mathrm{e}^{i\mathbf{r}\cdot\mathbf{k}} \langle \delta \mathbf{B}(\mathbf{r}_0) \delta \mathbf{B}(\mathbf{r}_0 + \mathbf{r}) \rangle \propto k^{-5/3}$$
- Pitch-angle scattering induces spatial diffusion:

$$\kappa \simeq \frac{c^2}{3\Gamma} \propto \frac{1}{\Omega} \left(\frac{k\, P(k)}{B_0^2} \right)^{-1} \propto p^{1/3}$$

Resonant interactions



Momentum diffusion

- Waves carry a momentum
- When scattering particles, can transfer momentum to CRs
- If angular distribution of waves is not isotropic, CRs get accelerated
- Stochastic process, can be described as diffusion in momentum space:

$$\frac{\partial f}{\partial t} - \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial f}{\partial p} \right) = 0$$
$$\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi \right) = 0$$

Momentum change

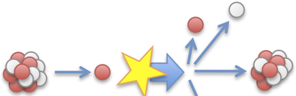
- CRs lose energy by interactions with radiation fields:
  - Synchrotron emission
  - Inverse Compton scattering
- . . . or gas:
  - Bremsstrahlung
  - Ionisation losses
  - Coulomb scattering
- Adiabatic losses or gains:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}t} \right)_{\mathrm{ad}} = -\frac{p}{3} \left(\nabla \cdot \mathbf{U} \right)$$

- In transport equation:

$$\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial p} \left(-\frac{\mathrm{d}p}{\mathrm{d}t} \psi + \frac{p}{3} \left(\nabla \cdot \mathbf{U} \right) \psi \right) = 0$$

Spallation and decay



- In leaky box model, considered a global rate, e.g.

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -\frac{N_2}{t_{\mathrm{esc}}} - N_2 \Gamma_2 + \Gamma_{1\to 2} N_1$$

- In a diffusion model

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} \supset -v n_{\mathrm{gas}} \sigma_j \psi_j + v n_{\mathrm{gas}} \sum_{k>j} \sigma_{k\to j} \psi_k$$

- Additionally, decay:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} \supset -\frac{\psi_j}{\tau_j} + \sum_{k>j} \frac{\psi_k}{\tau_{k\to j}}$$

The transport equation

$$\frac{\partial \psi_j}{\partial t} = \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U}\psi_j)$$
spatial diffusion and advection

$$+ \frac{\partial}{\partial p}\left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j\right)$$
momentum diffusion

$$+ \frac{\partial}{\partial p}\left(-\frac{\mathrm{d}p}{\mathrm{d}t}\psi_j + \frac{p}{3}(\nabla \cdot \mathbf{U})\,\psi_j\right)$$
momentum change incl. adiabatic

$$- v n_{\mathrm{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j}$$
spallation and decay

$$+ v n_{\mathrm{gas}} \sum_{k>j} \sigma_{k\to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k\to j}}$$
spallation and decay

$$+ q_j$$
primary sources

The transport equation

$$\frac{\partial \psi_j}{\partial t} = \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j)$$ spatial diffusion and advection

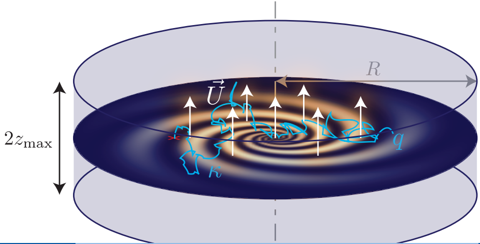
$$+ \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right)$$ momentum diffusion

$$+ \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} (\nabla \cdot \mathbf{U}) \psi_j \right)$$ momentum change incl. adiabatic

$$- v n_{\text{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j}$$ spallation and decay

$$+ v n_{\text{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}}$$ spallation and decay

$$+ q_j$$ primary sources



Leaky box model

- Can derive leaky box model from transport equation

$$\frac{\partial \psi_j}{\partial t} - \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j) = q_j + \sum_{j<k} (v n_{\mathrm{gas}} \sigma_{k \to j}) \, \psi_k - (v n_{\mathrm{gas}} \sigma_i) \, \psi_j$$

- Integrate over CR halo, $N_j \equiv \int \mathrm{d}V \, \psi_j$:

$$\frac{\partial N_j}{\partial t} - \int \mathrm{d}V \left[\nabla \cdot \underbrace{(\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j)}_{= -\mathbf{F}_j} \right] = \frac{\partial N_j}{\partial t} + \underbrace{\int \mathrm{d}\mathbf{S} \cdot \mathbf{F}_j}_{\equiv -\frac{N_j}{\tau_{\mathrm{esc}}}} = Q_j + \sum_{j<k} (v n_{\mathrm{gas}} \sigma_{k \to j}) \, N_k - (v n_{\mathrm{gas}} \sigma_i) \, N_j$$

Simplified 1D model

Simplify transport equation:

$$\frac{\partial \psi_j}{\partial t} - \kappa \frac{\partial^2 \psi_j}{\partial z^2} - U \frac{\partial \psi_j}{\partial z} + \left(v n_{\text{gas}}(z) \sigma_j + \frac{1}{\tau_j} \right) \psi_j = q_j + \sum_{j<k} \left(v n_{\text{gas}}(z) \sigma_{k \to j} + \frac{1}{\tau_{k \to j}} \right) \psi_k$$

$$-\kappa \frac{\partial^2 \psi_j}{\partial z^2} - U \frac{\partial \psi_j}{\partial z} + \left(2h\delta(z) v n_{\text{gas}} \sigma_j + \frac{1}{\tau_j} \right) \psi_j = 2h\delta(z) q_j + \sum_{j<k} \left(2h\delta(z) v n_{\text{gas}} \sigma_{k \to j} + \frac{1}{\tau_{k \to j}} \right) \psi_k$$

Solution

$$\psi_j(z, p) = \frac{2 z_{\max} \left(q_j + \sum_{k>j} v n_{\text{gas}} \sigma_{k \to j} \psi_k \right) \mathrm{e}^{\frac{zU}{2\kappa}} \sinh \left[\frac{z_{\max} - z}{z_j} \right]}{\left(2h v n_{\text{gas}} \sigma_j + U + \frac{\kappa}{z_{\max}} \frac{z_{\max}}{z_j} \coth \left[\frac{z_{\max}}{z_j} \right] \right) \sinh \left[\frac{z_{\max}}{z_j} \right]}$$

with $z_j \equiv \left(\left(\frac{U}{2\kappa} \right)^2 - \frac{1}{\tau_j \kappa^2} \right)^{-1/2}$

Simplified 1D model

Simplify transport equation:

$$\frac{\partial \psi_j}{\partial t} - \kappa \frac{\partial^2 \psi_j}{\partial z^2} - U \frac{\partial \psi_j}{\partial z} + \left(v n_{\mathrm{gas}}(z) \sigma_j + \frac{1}{\tau_j} \right) \psi_j = q_j + \sum_{j<k} \left(v n_{\mathrm{gas}}(z) \sigma_{k \to j} + \frac{1}{\tau_{k \to j}} \right) \psi_k$$

$$-\kappa \frac{\partial^2 \psi_j}{\partial z^2} - U \frac{\partial \psi_j}{\partial z} + \left(2h\delta(z) v n_{\mathrm{gas}} \sigma_j + \frac{1}{\tau_j} \right) \psi_j = 2h\delta(z) q_j + \sum_{j<k} \left(2h\delta(z) v n_{\mathrm{gas}} \sigma_{k \to j} + \frac{1}{\tau_{k \to j}} \right) \psi_k$$

Solution

$$\psi_j(0, p) = \frac{2 z_{\mathrm{max}} \left(q_j + \sum_{k>j} v n_{\mathrm{gas}} \sigma_{k \to j} \psi_k \right) \mathrm{e}^{\frac{zU}{2\kappa}} \sinh \left[\frac{z_{\mathrm{max}} - z}{z_j} \right]}{\left(2 h v n_{\mathrm{gas}} \sigma_j + U + \frac{\kappa}{z_{\mathrm{max}}} \frac{z_{\mathrm{max}}}{z_j} \coth \left[\frac{z_{\mathrm{max}}}{z_j} \right] \right) \sinh \left[\frac{z_{\mathrm{max}}}{z_j} \right]}$$

with $z_j \equiv \left(\left(\frac{U}{2\kappa} \right)^2 - \frac{1}{\tau_j \kappa^2} \right)^{-1/2} \to \infty$; evaluate at $z = 0$

$$\text{ambient CR spectrum} = \frac{\text{source spectrum}}{\text{diffusion coefficient}}$$

Green's function approach

- Solve simplified transport equation for $\mathrm{e}^{\pm}$

$$\frac{\partial \psi_{\pm}}{\partial t} - \nabla \cdot \kappa \cdot \nabla \psi_{\pm} - \frac{\partial}{\partial E} \left(\frac{\mathrm{d}E}{\mathrm{d}t} \psi_{\pm} \right) = \delta(\mathbf{r} - \mathbf{r}_0) \delta(t) Q(E)$$

- For homogeneous energy loss rate $\mathrm{d}E/\mathrm{d}t$, energy becomes pseudo time
- Heat equation

$$\Rightarrow \psi_{\pm}(\mathbf{r}, E, t) = \left(\pi \ell^2(E, t) \right)^{-3/2} \mathrm{e}^{-|\mathbf{r}-\mathbf{r}_0|^2/\ell^2(E,t)} \frac{\frac{\mathrm{d}E}{\mathrm{d}t}(E)}{\frac{\mathrm{d}E}{\mathrm{d}t}(E_0)} Q(E_0)$$

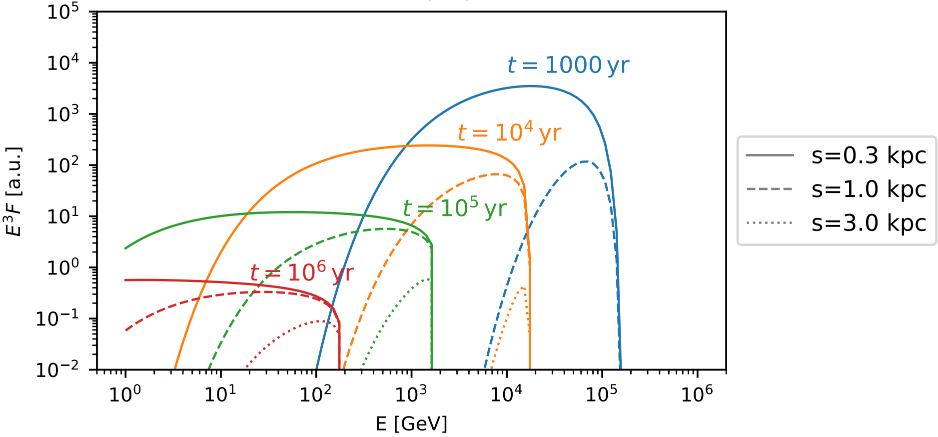
where $E_0 = E_0(E, t)$ and

$$\ell^2(E, t) = 4 \int_{E_0}^{E} \mathrm{d}E' \, \frac{\kappa(E')}{\mathrm{d}E/\mathrm{d}t}$$

Green's function

$$\psi_\pm(\mathbf{r}, E, t) = \left(\pi \ell^2(E,t)\right)^{-3/2} \mathrm{e}^{-|\mathbf{r}-\mathbf{r}_0|^2/\ell^2(E,t)} \frac{\frac{\mathrm{d}E}{\mathrm{d}t}(E)}{\frac{\mathrm{d}E}{\mathrm{d}t}(E_0)} Q\left(E_0\right)$$

Mertsch (2018)



with $\kappa_{xx} = \kappa_0 E^\delta$, $\frac{\mathrm{d}E}{\mathrm{d}t} - b_0 E^2$, $Q(E_0) \propto E_0^{-\Gamma} \exp[-E_0/E_c]$

The cosmic-ray transport equation

$$\frac{\partial \psi_j}{\partial t} = \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j)$$ spatial diffusion and advection

$$+ \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right)$$ momentum diffusion

$$+ \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} (\nabla \cdot \mathbf{U}) \psi_j \right)$$ momentum change incl. adiabatic

$$- v n_{\mathrm{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j}$$ spallation and decay

$$+ v n_{\mathrm{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}}$$ spallation and decay

$$+ S_j$$ primary sources

Idealised setup

- Isotropy
- Gaussianity
- ...

The cosmic-ray transport equation

$$\frac{\partial \psi_j}{\partial t} = \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j)$$ spatial diffusion and advection

$$+ \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right)$$ momentum diffusion

$$+ \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} (\nabla \cdot \mathbf{U}) \psi_j \right)$$ momentum change incl. adiabatic

$$- v n_{\text{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j}$$ spallation and decay

$$+ v n_{\text{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}}$$ spallation and decay

$$+ S_j$$ primary sources

Idealised setup

- Isotropy
- Gaussianity
- ...

Frequent assumptions

- Symmetry
- Ignorable dimensions
- Smooth source distribution

In reality

- No exact symmetries
- 3D
- Point-like sources

$\rightarrow$ Numerical solution needed

Global fit

Large number of free parameters

- Unknown parameters:

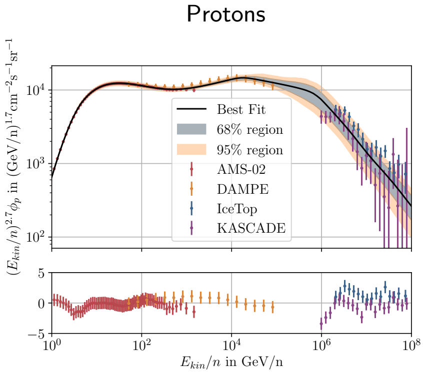
$$\left.\begin{array}{l}\blacksquare\ \text{Source spectrum: } \gamma^{\mathrm{p}},\ \gamma^{\mathrm{He}},\ \gamma^{\mathrm{C}};\ N_{\mathrm{He}},\ N_{\mathrm{C}};\ (e^{\pm}\text{ parameters}) \\ \blacksquare\ \text{Gal. transport: } \kappa_0;\ \delta_1\ldots\delta_5;\ \mathcal{R}_{12}\ldots\mathcal{R}_{45};\ s_{12}\ldots s_{45} \\ \blacksquare\ \text{Solar modulation: } \phi_{\mathrm{p}},\ \phi_{\mathrm{e^+}},\ \phi_{\mathrm{e^-}},\ \phi_{\mathrm{nuc}} \\ \blacksquare\ \text{Energy scale uncert.: } \alpha_{\text{AMS-02}},\ \alpha_{\text{DAMPE}},\ \alpha_{\text{IceTop}},\ \alpha_{\text{KASCADE}}\end{array}\right\}\mathcal{O}(30)$$

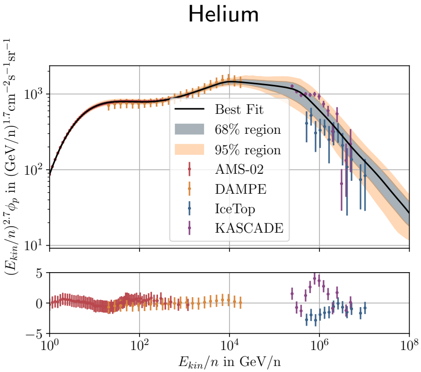
$\rightarrow$ Need to efficiently scan parameter space

- Used affine-invariant MC sampler `emcee` Foreman-Mackey *et al.* (2012)

Cosmic ray results

Schwefer, Mertsch, Wiebusch (2022)

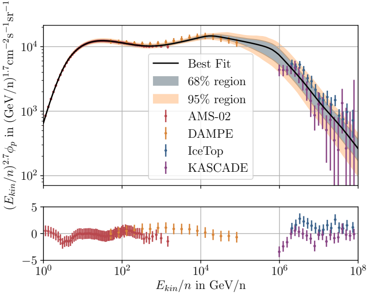


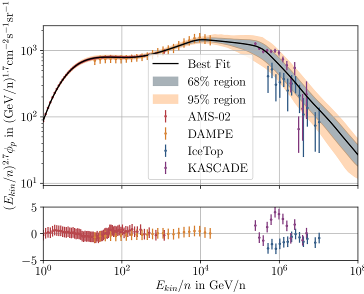


Anatomy of a diffusion coefficient

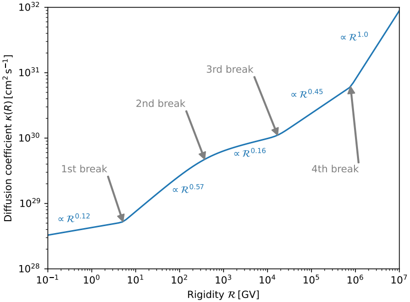
Schwefer, Mertsch, Wiebusch (2022)

Proton





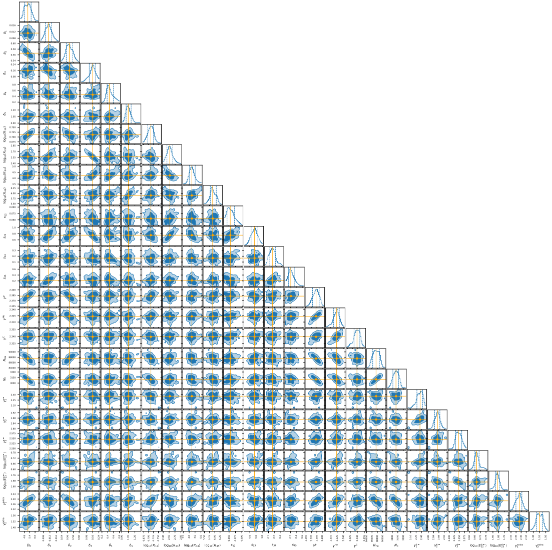
Helium



- 1. Break: Hardening of spectra at low energies due to stochasticity effect Phan *et al.* (2021)
- 2. Break: Transition from self-generated to external turbulence Blasi, Amato, Serpico (2012)
- 3. Break: ?!
- 4. Break: Transition from resonant to small-angle scattering

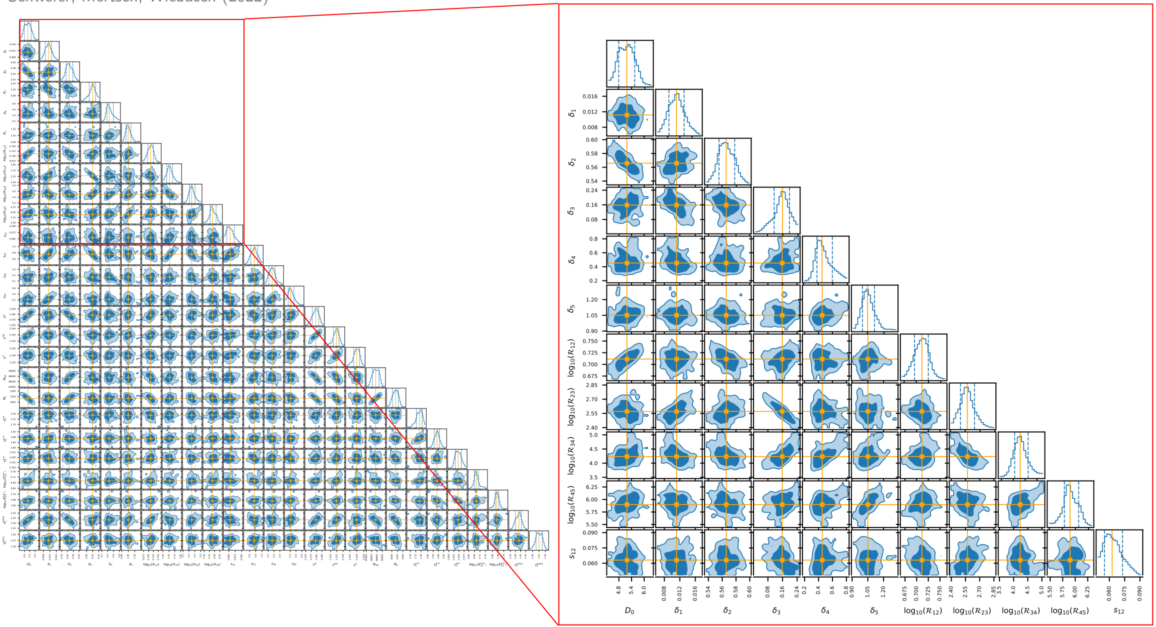
Challenges

Schwefer, Mertsch, Wiebusch (2022)



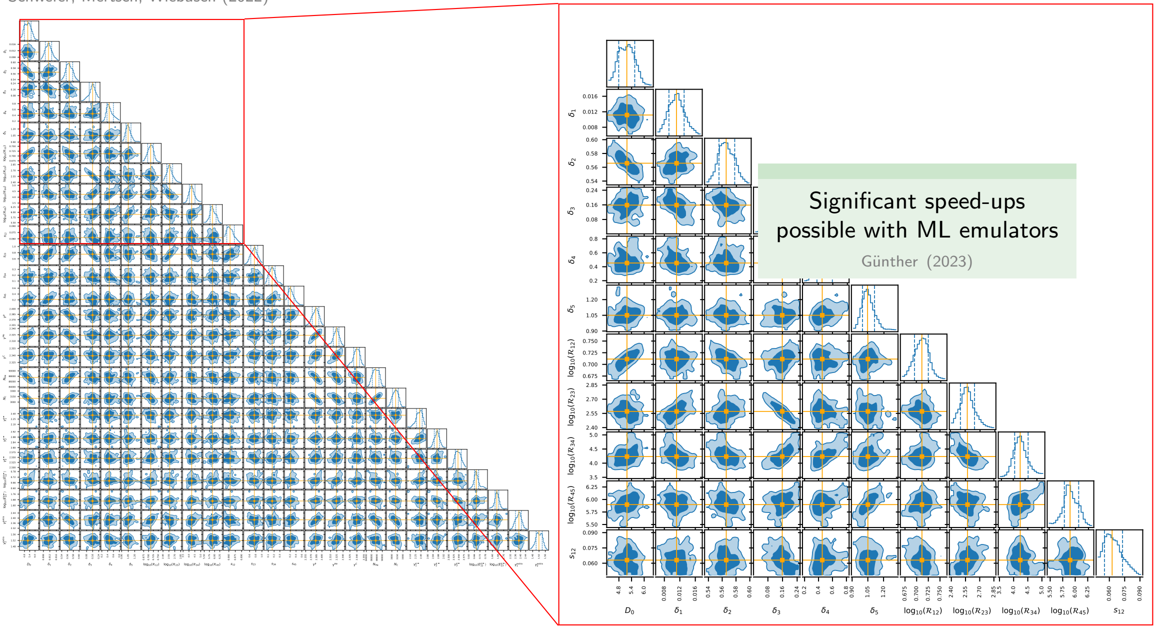
Challenges

Schwefer, Mertsch, Wiebusch (2022)



Challenges

Schwefer, Mertsch, Wiebusch (2022)



Significant speed-ups possible with ML emulators

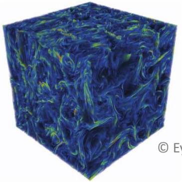
Günther (2023)

Beyond quasi-linear theory

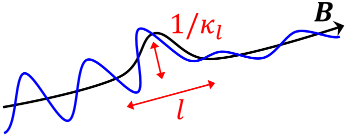
Lemoine, Kempski, . . .

Magneto-hydrodynamic turbulence
→ anisotropic power spectra
→ efficiency of scattering reduced

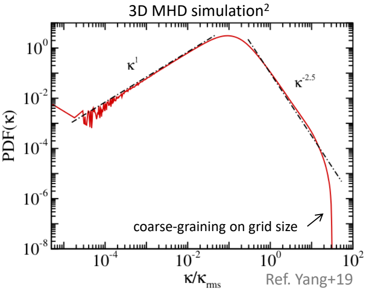
Chandran, Cho, Lazarian, Yang, . . .



Scattering with non-wave-like turbulence



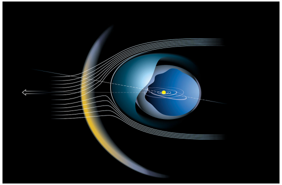
Magnetic field-line curvature scattering
→ mean-free path comparable to quasi-linear theory

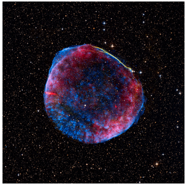
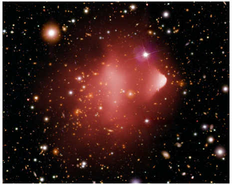


Outline

Shocks are ubiquitous

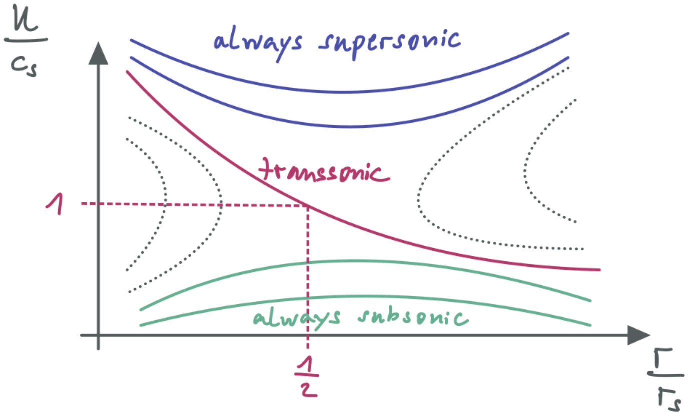
- A flow is characterised by ρ, $\mathbf{U}$, T
- In general, functions of position
- If they change abruptly, this is called a shock
- Shocks are very common in astrophysics:
  - Blast wave explosion
  - Accretion (onto star, black hole, galaxy cluster)
  - Mergers
  - Supersonic winds / outflows
  - Bow shocks (=supersonic flow around obstacle)



Shocks are inevitable

- SN is expelling material at $U_{\mathrm{sh}} > c_s$
- Supersonic flow close to SN, subsonic flow at infinity
- Nature of supersonic and subsonic flows very different



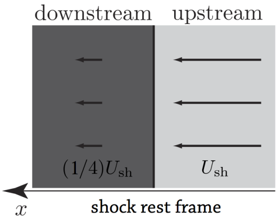
$\rightarrow$ Connection must be discontinuous: formation of shock

Parallel shock

- Consider non-relativistic shock in its rest frame
- Discontinuity in gas density and velocity:

$$\rho_2 = r\rho_1 \quad \text{and} \quad U_2 = \frac{1}{r} U_1$$

→ Gas is compressed and slowed down



Compression ratio

Depends on the ratio of specific heats γ:

$$r \simeq \frac{\gamma + 1}{\gamma - 1}$$

For ideal mono-atomic gas: $\gamma = 5/3 \Rightarrow r = 4$

The transport equation

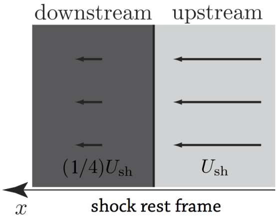
$$
\begin{aligned}
\frac{\partial \psi_j}{\partial t} = & \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j) && \text{spatial diffusion and advection} \\
& + \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right) && \text{momentum diffusion} \\
& + \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} (\nabla \cdot \mathbf{U}) \psi_j \right) && \text{momentum change incl. adiabatic} \\
& - v n_{\text{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j} && \text{spallation and decay} \\
& + v n_{\text{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}} && \text{spallation and decay} \\
& + S_j && \text{primary sources}
\end{aligned}
$$

The transport equation

$$
\begin{aligned}
\frac{\partial \psi_j}{\partial t} = & \nabla \cdot (\kappa \cdot \nabla \psi_j - \mathbf{U} \psi_j) && \text{spatial diffusion and advection} \\
& + \frac{\partial}{\partial p} \left(p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi_j \right) && \text{momentum diffusion} \\
& + \frac{\partial}{\partial p} \left(- \frac{\mathrm{d}p}{\mathrm{d}t} \psi_j + \frac{p}{3} (\nabla \cdot \mathbf{U}) \psi_j \right) && \text{momentum change incl. adiabatic} \\
& - v n_{\text{gas}} \sigma_j \psi_j - \frac{\psi_j}{\tau_j} && \text{spallation and decay} \\
& + v n_{\text{gas}} \sum_{k>j} \sigma_{k \to j} \psi_k + \sum_{k>j} \frac{\psi_k}{\tau_{k \to j}} && \text{spallation and decay} \\
& + S_j && \text{primary sources}
\end{aligned}
$$

Macroscopic approach

Seminal papers in 1977/78 by Krymsky; Axford, Leer, Skaldron; Blandford, Ostriker



- Consider steady-state transport equation for phase-space density f:

$$U \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \kappa \frac{\partial f}{\partial x} - \frac{p}{3} \frac{\mathrm{d}U}{\mathrm{d}x} \frac{\partial f}{\partial p} = 0$$

- For $x \neq 0$,

$$f(x,p) = \begin{cases} g_1(p) \exp\left[\frac{x}{\kappa(p)/U}\right] + f_1(p) & \text{for } x < 0 \\ f_2(p) & \text{for } x > 0 \end{cases}$$

Macroscopic approach

Seminal papers in 1977/78 by Krymsky; Axford, Leer, Skaldron; Blandford, Ostriker

- Can derive matching conditions and find for the spectrum at shock,

$$f_2(p) = \Gamma p^{-\Gamma} \int_0^p \mathrm{d}p' \, p'^{\Gamma-1} f_1(p') + \text{const.} \times p^{-\Gamma}$$

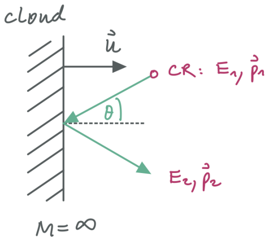
  with spectral index $\Gamma \equiv \frac{3r}{r-1}$
- With $r \simeq \frac{\gamma+1}{\gamma-1} = 4$, $f(0,p) \propto p^{-4} \Rightarrow \psi(0,p) = 4\pi p^2 f(0,p) \propto p^{-2}$

Strong ($r = 4$) shock accelerates CRs to p^{-2} spectrum!

Microscopic approach

Fermi (1949)

- Collision of particle with ∞ massive cloud



- Two types of collisions:
  - $\theta \in [0, \frac{\pi}{2}]$ "head-on"
  - $\theta \in [\frac{\pi}{2}, \pi]$ "trailing"

1. Transform E and p_x to cloud frame (primed):

$$E_1' = \gamma(E_1 + U p_1 \cos\theta)$$

$$p_{1x}' = p_1' \cos\theta' = \gamma\left(p_1 \cos\theta + \frac{U}{c^2} E_1\right)$$

Microscopic approach

1. Transform E and p_x to cloud frame (primed):

$$E_1' = \gamma(E_1 + U p_1 \cos\theta)$$

$$p_{1x}' = p_1' \cos\theta' = \gamma\left(p_1 \cos\theta + \frac{U}{c^2} E_1\right)$$

2. Collision in cloud frame:
   - Energy conserved: $E_2' = E_1'$
   - Momentum flipped: $p_{2x}' = -p_{1x}'$
3. Transform back to observer frame:

$$\begin{aligned}
E_2 &= \gamma\left(E_2' - U p_{2x}'\right) \\
&= \gamma\left(\gamma\left(E_1 + U p_1 \cos\theta\right) + U\gamma\left(p_1 \cos\theta + \frac{U}{c^2} E_1\right)\right) \\
&= \gamma^2\left(E_1 + 2 U p_1 \cos\theta + E_1 \left(\frac{U}{c}\right)^2\right) \\
&= \gamma^2 E_1 \left(1 + 2U \frac{v_1}{c^2} \cos\theta + \left(\frac{U}{c}\right)^2\right)
\end{aligned}$$

Microscopic approach

3. Transform back to observer frame:

$$E_2 = \gamma^2 E_1 \left(1 + 2U \frac{v_1}{c^2} \cos\theta + \left(\frac{U}{c} \right)^2 \right)$$

4. Expand γ^2 in U/c

$$\gamma^2 = \left(1 - (U/c)^2 \right)^{-1} \simeq 1 + (U/c)^2$$

and let $v_1 \simeq c$:

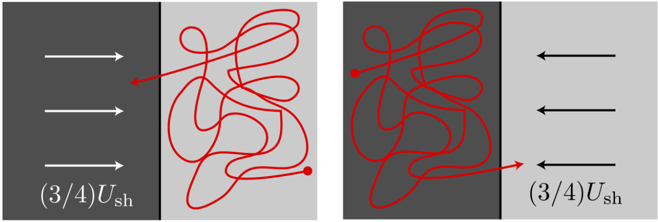
$$\Rightarrow E_2 \simeq E_1 \left(1 + 2\frac{U}{c} \cos\theta + 2\left(\frac{U}{c} \right)^2 \right)$$

$$\Rightarrow \Delta E \equiv E_2 - E_1 = E_1 \left(2\frac{U}{c} \cos\theta + 2\left(\frac{U}{c} \right)^2 \right)$$

$$\begin{aligned} \cos\theta > 1 \quad &\Rightarrow \quad \text{energy gain} \\ \cos\theta < 1 \quad &\Rightarrow \quad \text{energy loss} \end{aligned}$$

Microscopic approach

Seminal papers in 78 by Bell

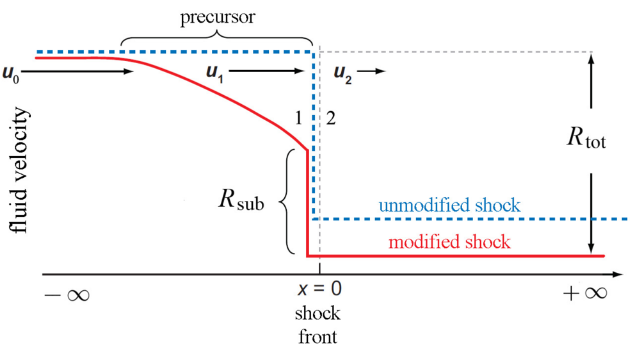


upstream rest frame   downstream rest frame

- Particle crossing shock always suffers "head-on" collisions

$\rightarrow$ Systematic energy gain, $\Delta E/E \sim (U_{\rm sh}/c)$

Non-linear shock acceleration



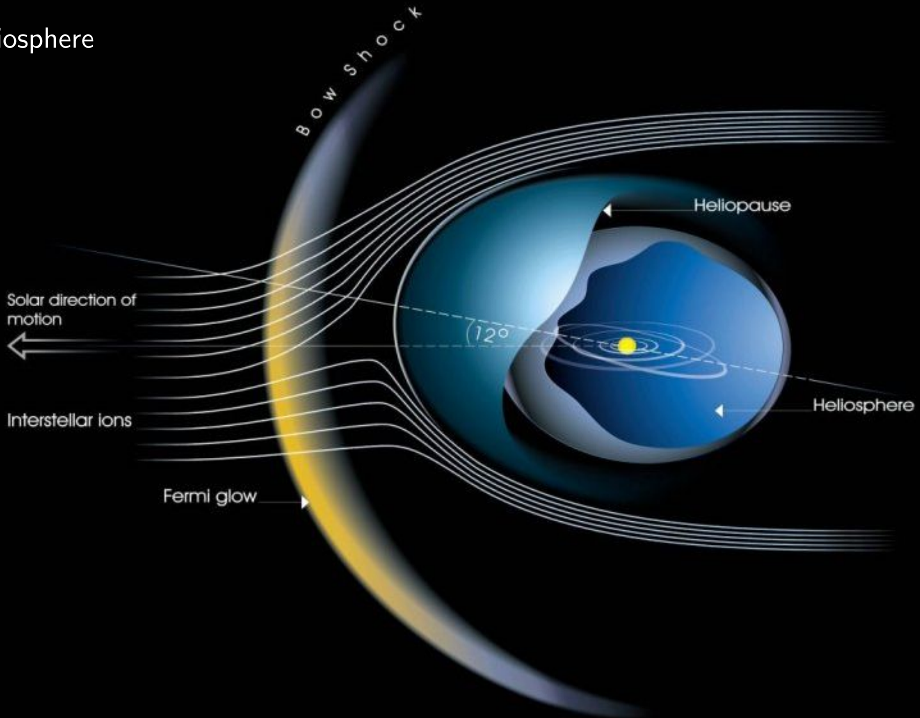
Shock modification

- CRs contribute to the pressure of the system
- Slow-down of flow $\rightarrow$ precursor
- Compression depends on position

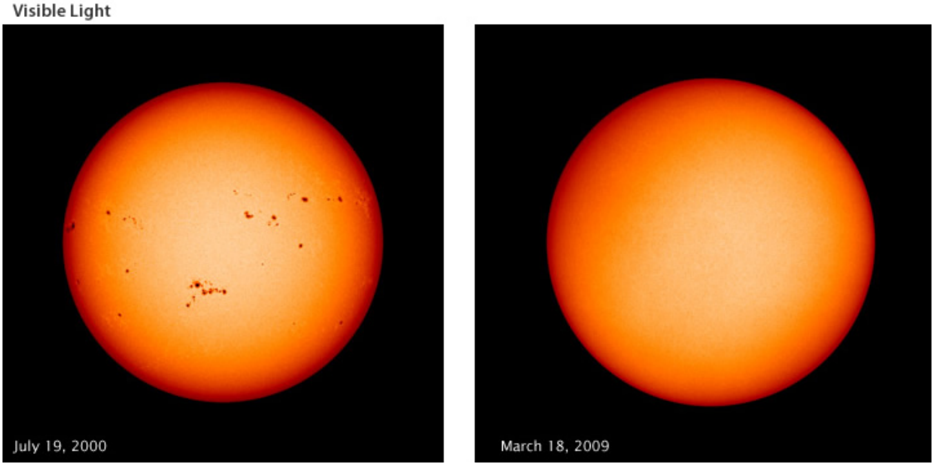
$\rightarrow$ Spectral curvature

Outline

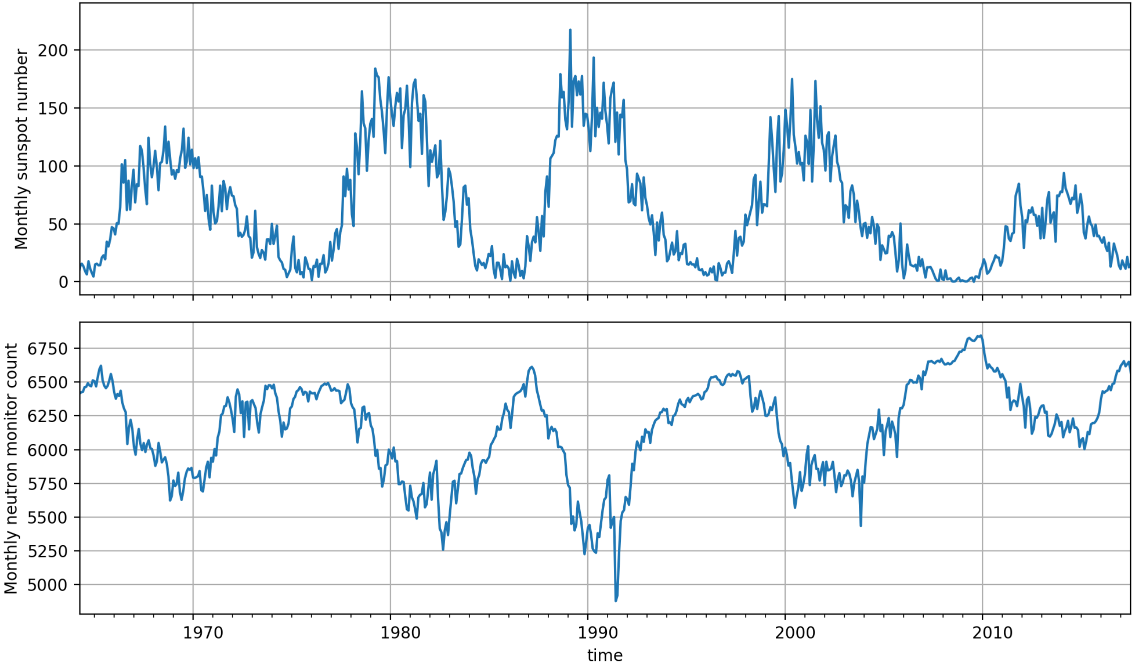
Heliosphere



Solar activity



Solar activity



Force-field approximation

- Transport equation:

$$\frac{\partial f}{\partial t} - \nabla \cdot (\kappa \cdot \nabla f - C\mathbf{v}f) + \frac{1}{3p^2}\frac{\partial}{\partial p}\left(p^3\mathbf{v} \cdot \nabla f\right) = q$$

with Compton Getting factor $C \equiv -\frac{1}{3}\frac{\partial \ln f}{\partial \ln p}$

Assumptions

- Steady state: $\partial f / \partial t = 0$
- No sources: $q = 0$
- No momentum loss in fixed frame: $\left\langle \frac{\mathrm{d}p}{\mathrm{d}t} \right\rangle = \frac{1}{3}\mathbf{v} \cdot \frac{\nabla f}{f} = 0$

$\Rightarrow$ Zero-streaming condition:

$$\kappa \cdot \nabla f - C\mathbf{v}f = 0$$

Force-field solution

Gleeson & Axford, Gleeson & Urch, Caballero-Lopez & Moraal

- Spherical symmetry:

$$\frac{\partial f}{\partial r} + \frac{Vp}{3\kappa}\frac{\partial f}{\partial p} = 0$$

- Solved by method of characteristics:

$$f_{\text{TOA}}(p_{\text{TOA}}) = f_{\text{LIS}}(p_{\text{LIS}})$$

TOA: top of the atmosphere, LIS: local interstellar

- Assume factorisation, $\kappa(r, p) = \kappa_0(r) p$

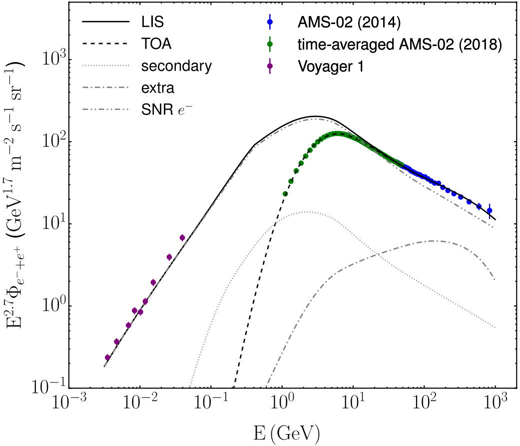
$\Rightarrow$ $p_{\text{TOA}} = p_{\text{LIS}} - \varphi$ where

$$\varphi = \int_{r_{\text{TOA}}}^{r_{\text{LIS}}} \mathrm{d}r' \, \frac{v}{3\kappa_0(r')}$$

- Simple relation

$$J_{\text{TOA}}(p_{\text{TOA}}) = \frac{(p_{\text{TOA}})^2}{(p_{\text{TOA}} + \varphi)^2} J_{\text{LIS}}(p_{\text{TOA}} + \varphi)$$

Vittino, Mertsch, Gast, Schael



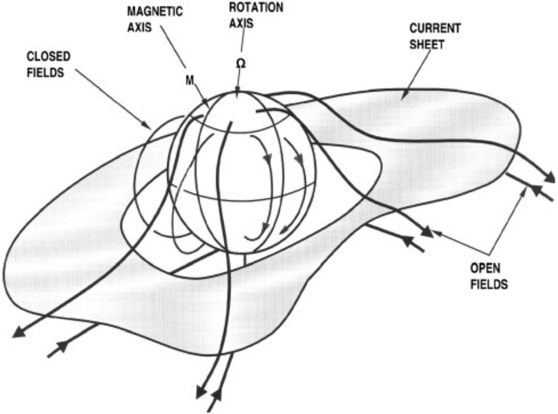
Drift effects

- In inhomogeneous **B**-fields, CRs move $\perp$ to field lines:

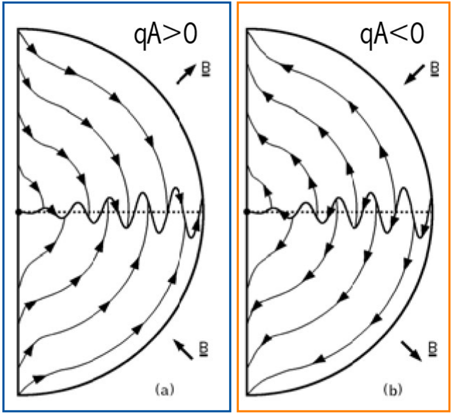
$$\frac{\partial f}{\partial t} - \nabla \cdot (\kappa \cdot \nabla f - C\mathbf{v} f - \mathbf{v}_\mathrm{D} f) + \frac{1}{3p^2} \frac{\partial}{\partial p} \left(p^3 \mathbf{v} \cdot \nabla f \right) = q$$

where $\mathbf{v}_\mathrm{D} \propto \nabla \times \hat{\mathbf{b}}$ depends on charge

- Also: current-sheet drifts



Charge-sign dependent modulation



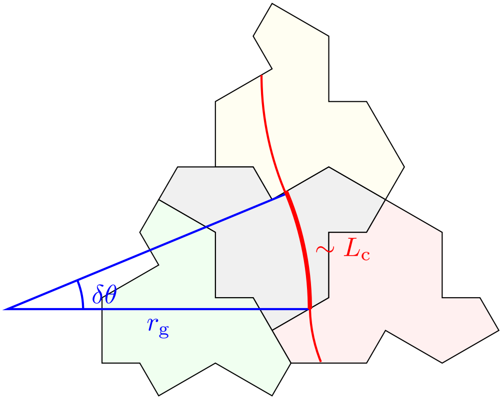
Drift patterns of positively charged particles

- In $A > 0$ cycles, $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ particles arrive mostly from $\left\{\begin{matrix}\text{polar}\\ \text{equatorial}\end{matrix}\right\}$ direction
- In $A < 0$ cycles, $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ particles arrive mostly from $\left\{\begin{matrix}\text{equatorial}\\ \text{polar}\end{matrix}\right\}$ direction

$\rightarrow$ Different propagation times $\rightarrow$ different energy losses $\rightarrow$ different modulation

Any questions?

Small-angle scattering



- Mean-square deflection from crossing one domain:

$$\langle (\delta\theta)^2 \rangle \simeq \left(\frac{L_\mathrm{c}}{r_\mathrm{g}} \right)^2$$

- Number of domains crossed in time Δt:

$$N \simeq \frac{c \Delta t}{L_\mathrm{c}}$$

- Mean-square deflection from crossing N domains:

$$\langle (\Delta\theta)^2 \rangle = N \langle (\delta\theta)^2 \rangle \simeq \frac{c \Delta t}{L_\mathrm{c}} \left(\frac{L_\mathrm{c}}{r_\mathrm{g}} \right)^2$$

- When $\langle (\Delta\theta)^2 \rangle = 1$, $\Delta t \equiv t_\mathrm{sc}$:

$$t_\mathrm{sc} = \frac{L_\mathrm{c}}{c} \left(\frac{r_\mathrm{g}}{L_\mathrm{c}} \right)^2$$

- Spatial diffusion coefficient:

$$\kappa = \frac{c^2}{3} t_\mathrm{sc} = \frac{c L_\mathrm{c}}{3} \left(\frac{r_\mathrm{g}}{L_\mathrm{c}} \right)^2$$

We do not understand how supernova remnants accelerate to E_{knee}

What is $E_{\max}$?

- Equate age with acceleration time: $t_{\mathrm{age}} = t_{\mathrm{acc}} = 8 \frac{\kappa}{U_{\mathrm{sh}}^2}$ Diffusion coefficient

- Assume Bohm diffusion: $\kappa = \frac{c \ell_{\mathrm{mfp}}}{3} = \frac{c r_{\mathrm{g}}}{3} = \frac{c}{3} \frac{E_{\max}}{qB}$ Gyro radius

- Hillas-like relation: $\Rightarrow E_{\max} \simeq \frac{U_{\mathrm{sh}}^2}{c} qB t_{\mathrm{age}}$ or $\frac{U_{\mathrm{sh}}}{c} qBR$

We do not understand how supernova remnants accelerate to E_{knee}

What is E_{max}?

- Equate age with acceleration time: $$t_{\mathrm{age}} = t_{\mathrm{acc}} = 8\frac{\kappa}{U_{\mathrm{sh}}^2}$$ Diffusion coefficient

Gyro radius

- Assume Bohm diffusion: $$\kappa = \frac{c\ell_{\mathrm{mfp}}}{3} = \frac{c r_{\mathrm{g}}}{3} = \frac{c}{3}\frac{E_{\mathrm{max}}}{qB}$$
- Hillas-like relation: $$\Rightarrow E_{\mathrm{max}} \simeq \frac{U_{\mathrm{sh}}^2}{c} q B t_{\mathrm{age}} \quad \text{or} \quad \frac{U_{\mathrm{sh}}}{c} q B R$$

- With typical values: $$U_{\mathrm{sh}} = 10^4\,\mathrm{km\,s^{-1}}\,, \quad B = 1\,\mu\mathrm{G}\,, \quad t_{\mathrm{age}} = 10^3\,\mathrm{yr}$$

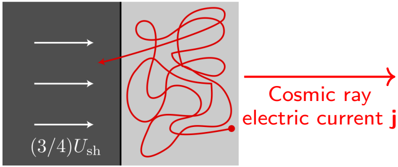
$$\Rightarrow E_{\mathrm{max,b}} \simeq 100\,\mathrm{TeV} \ll E_{\mathrm{knee}}$$

Lagage and Cesarsky (1983)

1. Choosing larger t_{age} does not help: $U_{\mathrm{sh}} \propto t_{\mathrm{age}}^{-3/5}$, so E_{max} decreases with time
2. Need to amplify **B**-field to $B \simeq 100\,\mu\mathrm{G}$

The Bell instability can amplify B-fields

Bell (2004)



upstream rest frame

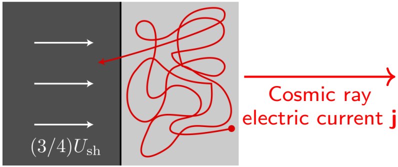
- If B-field too weak, particles escape

$\rightarrow$ Electric current $\mathbf{j}$

- Waves modes unstable in the presence of current $\mathbf{j}$

The Bell instability can amplify B-fields

Bell (2004)

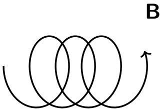


upstream rest frame

- If B-field too weak, particles escape

$\rightarrow$ Electric current $\mathbf{j}$

- Waves modes unstable in the presence of current $\mathbf{j}$



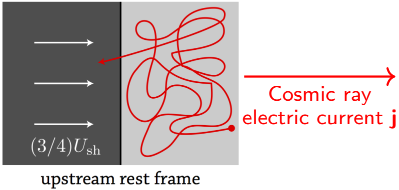
- CRs with gyroradius r_g tied to field lines

$\rightarrow$ Instability saturates once $\lambda \sim r_g$

- **B**-field density satisfies $\varepsilon_B \sim \dfrac{U_{sh}}{c} \varepsilon_{CR}$

The Bell instability can amplify B-fields

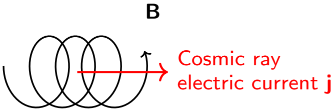
Bell (2004)



- If B-field too weak, particles escape

→ Electric current **j**

- Waves modes unstable in the presence of current **j**



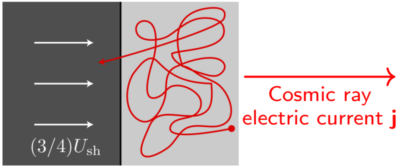
- CRs with gyroradius $r_{\rm g}$ tied to field lines

→ Instability saturates once $\lambda \sim r_{\rm g}$

- **B**-field density satisfies $\varepsilon_B \sim \frac{U_{\rm sh}}{c} \varepsilon_{\rm CR}$

The Bell instability can amplify B-fields

Bell (2004)

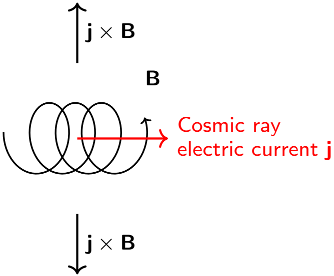


upstream rest frame

- If B-field too weak, particles escape

$\rightarrow$ Electric current **j**

- Waves modes unstable in the presence of current **j**



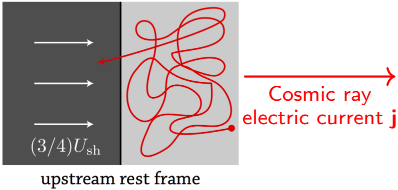
- CRs with gyroradius r_g tied to field lines

$\rightarrow$ Instability saturates once $\lambda \sim r_g$

- **B**-field density satisfies $\varepsilon_B \sim \dfrac{U_{\text{sh}}}{c} \varepsilon_{\text{CR}}$

The Bell instability can amplify B-fields

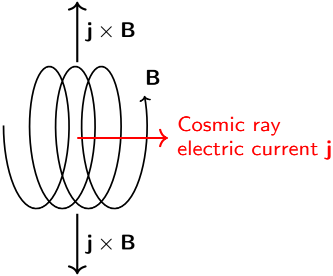
Bell (2004)



- If B-field too weak, particles escape

→ Electric current **j**

- Waves modes unstable in the presence of current **j**



- CRs with gyroradius r_g tied to field lines

→ Instability saturates once $\lambda \sim r_g$

- **B**-field density satisfies $\varepsilon_B \sim \frac{U_{\mathrm{sh}}}{c} \varepsilon_{\mathrm{CR}}$

The highest CR energies can be achieved at start of Sedov-Taylor phase

- Shock speed $U_{\rm sh}$ enters into growth rate γ through escape current j
- Saturation field $B \propto U_{\rm sh}^{3/2}$
- $U_{\rm sh} \propto t_{\rm age}^{-3/5}$

$\rightarrow$ $E_{\rm max} \propto U_{\rm sh}^2 B t_{\rm age} \propto t_{\rm age}^{-11/10}$

Caprioli, Blasi, Amato (2009)